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I. + II. sem.

Matematika I

Zbirka riješenih zadataka za studente tehničkih fakulteta

Predgovor

Ova zbirka namjenjena je prvenstveno studentima Elektrotehničkog fakulteta, kao i studentima ostalih tehničkih fakulteta koji slušaju predmet „Matematika I" ili „Analiza I". Materijal korišten u izradi zbirke, najviše se naslanja na predavanja i vježbe iz predmeta „Matematika I" na Elektrotehničkom fakultetu, a obuhvata granične procese, nizove, redove, funkcije, integrale i uvodno poglavlje o funkcijama više promjenljivih. Osim ispitnih zadataka sadrži i teoretske osnove koje pomažu razumjevanju gradiva, kao i one zadatke koji su jako jednostavni, ali neophodni za savladivanje ispitnih zadataka. Preporučujem da se uz zbirku koristi i ostala literatura, i naglašavam da samo ova zbirka nije dovoljna da vam obezbjedi kompletnu satisfakciju kada ugledate ispitne rezultate.

Zahvaljujem se Heljić Jasminu, Fatkić Amiru, Gaguli Alemu koji su ukazali na neke greške u zbirci, kao i Avdibegović Dženanu koji je osim toga pružio nesebičnu materijalnu podršku, a pogotovo mojoj Almi bez koje ne bih bio u stanju završiti zbirku.

Autor

Napomena:

Pojedini zadaci su u svome izvornom obliku, u potpunosti preuzeti iz nekih knjiga koje su navedene u literaturi, iz prostog razloga da se studentu nađu pri ruci dok sprema ispitne zadatke, te ga oslobode traženja po ostaloj literaturi. Kako sam ja pri spremanju ispita koristio sve navedene knjige, toplo ih preporućujem svakom studentu.

SADRŽAJ

- Limes
- Nizovi ✓
- Redovi
- Funkcije ✓
- Integrali
- Funkcije više promjenljivih

1. Limes

1. $\lim\limits_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{n+3}$

2. $\lim\limits_{x\to 0}\left(\frac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}}$

3. $\lim\limits_{x\to 0}\left(\frac{\operatorname{arctg}x}{x}\right)^{\frac{1}{x^2}}$

4. $\lim\limits_{x\to 0}\frac{e^x-e^{\operatorname{tg}x}}{x-\operatorname{tg}x}$

5. $\lim\limits_{x\to 0}\frac{1-\sqrt{\cos x}\cdot\sqrt[3]{\cos 5x}}{x^2}$

6. $\lim\limits_{x\to 0_+} x^2\cdot\ln^3 x$

7. $f(x)=\frac{1}{1+2^{1/x}}$

8. $\lim\limits_{x\to+\infty} e^x\sin x$

9. $\lim\limits_{x\to+\infty}(x+\sin x)$

10. $y=x\sqrt{\frac{x-1}{x+1}}$

11. $\lim\limits_{x\to 0}\frac{x^2-1}{2x^2-x-1}$

12. $\lim\limits_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2+x^5}$

13. $\lim\limits_{x\to+\infty}\frac{(x+1)(x^2+1)\cdot\ldots\cdot(x^n+1)}{\left((nx)^n+1\right)^{\frac{n+1}{2}}}$

14. $\lim\limits_{x\to -1}\frac{x^3-2x-1}{x^5-2x-1}$

15. $\lim\limits_{x\to+16}\frac{\sqrt[4]{x}-2}{\sqrt{x}-4}$

16. $\lim\limits_{x\to 8}\frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2}$

17. $\lim\limits_{x\to 1}(1-x)\operatorname{tg}\frac{\pi x}{2}$

18. $\lim\limits_{x\to+\infty}\frac{\ln(2+e^{3x})}{\ln(3+e^{2x})}$

19. $\lim\limits_{n\to+\infty}\cos^n\frac{x}{\sqrt{n}}$

20. $\lim\limits_{x\to 0_+}\sqrt[x]{\cos\sqrt{x}}$

21. $f(x)=\frac{x(x^2-1)}{(x^2+1)(x-1)}$

22. $f(x,\alpha)=\begin{cases}\alpha k+1, & x<1\\ \sin x\pi+2, & x>1\end{cases}$

23. $f(x)=\frac{1}{1+2^{1/x}}$

24. $f(x)=\sin\frac{1}{x}$

25. $f(x,\alpha)=\begin{cases}\alpha x^2+x+3, & x\le 1\\ \sin\frac{\pi x}{2}+2\alpha, & x>1\end{cases}$

26. $f(x)=x^{n-1}\sin\frac{1}{x},\ x\neq 0$
$f(0)=0$

27. $f(x)=\ln\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}},\ x\neq 0$
$f(0)=\alpha$

28. $f(x)=x^2$
$x_0=1$

29. $f(x)=\ln x$

30. $y=\ln\left(\arcsin\frac{x}{x+1}\right)$

31. $\lim\limits_{x\to+\infty}\frac{4x-3}{2x+1}$

32. $\lim\limits_{x\to 0_+}\frac{e^{a^x}}{x^k}=L,\ k>0,\ a\neq 0$

33. $\lim\limits_{x\to+\infty}\frac{500x+10^4}{0{,}001x^2+1}$

34. $\lim\limits_{x\to -1}\frac{1+\sqrt[5]{x}}{1+\sqrt[3]{x}}$

35. $\lim\limits_{x\to 0}\frac{\sqrt[m]{1+x\sin 3x}\cdot\sqrt[n]{1+\operatorname{tg}2x}-1}{e^{\sin x}-1}$

36. $f(x) = 1 - \cos x, \; x \to 0$

37. $\varphi(x) = \operatorname{sh} x - x, \; x \to 0$

38. $\lim\limits_{x \to 0} \dfrac{\sqrt{1-\operatorname{tg} x} - \sqrt{1+\operatorname{tg} x}}{\sin 2x}$

39. $\lim\limits_{x \to 0} \dfrac{\sqrt[3]{1-\operatorname{tg} x} - \sqrt[3]{1+\operatorname{tg} x}}{\sin 3x}$

40. $\lim\limits_{x \to +\infty} \left(\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right)$

41. $\lim\limits_{x \to +\infty} \left(\sin\sqrt{x+1} - \sin\sqrt{x} \right)$

42. $\lim\limits_{x \to 1} (1-x) \operatorname{tg} \dfrac{\pi x}{2}$

43. $\lim\limits_{x \to +\infty} \dfrac{x^2}{e^x}$

44. $\lim\limits_{x \to +\infty} \dfrac{\ln^2 x}{x}$

45. $\lim\limits_{x \to 0} \dfrac{\cos x e^{x} - \cos x e^{-x}}{x^3}$

46. $\lim\limits_{x \to \infty} x^2 \ln \dfrac{\frac{1}{x}}{\sin \frac{1}{x}}$

47. $\lim\limits_{x \to 0} \left(\dfrac{\arcsin x}{x} \right)^{\frac{1}{x^2}}$

48. $\lim\limits_{x \to 0} \dfrac{\cos x \cos 2x \cdot \ldots \cdot \cos nx - 1}{x^2}$

49. $\lim\limits_{x \to \infty} \left(ax + b - \dfrac{x^3 - 1}{x^2 + 1} \right)$

50. $f_1(x) = \sqrt[n]{1+x} - 1, \; x \to 0$
$f_2(x) = \dfrac{x}{n}, \; n \in \mathbb{N}$

51. $f(x) = \sqrt{1-2x} - \sqrt[3]{1-3x}, \; x \to 0$

52. $f(x) = \operatorname{tg} x - \sin x, \; x \to 0$

53. $\lim\limits_{x \to 0} \dfrac{\cos(a+2x) - 2\cos(a+x) + \cos a}{x^2}$

54. $\lim\limits_{x \to \infty} \left(\dfrac{x-2}{2x+1} \right)^{x^2}$

55. $\lim\limits_{x \to \infty} \left(\dfrac{x^2+1}{x^2-2} \right)^{x^2}$

56. $\lim\limits_{x \to 0} \dfrac{\ln \cos ax}{\ln \cos bx}$

57. $\lim\limits_{x \to 0} \dfrac{1 - \cos^N x}{x^2}, \; N \neq 0$

58. $\lim\limits_{x \to 0} \dfrac{\operatorname{sh}^2 x}{\ln(\operatorname{ch} 3x)}$

59. $\lim\limits_{x \to \infty} \ln(1+2^x) \cdot \ln\left(1 + \dfrac{3}{x}\right)$

60. $\lim\limits_{n \to \infty} \sin \pi \sqrt{n^2+1}$

61. $\lim\limits_{n \to \infty} \prod\limits_{k=1}^{n} \left(1 + \dfrac{k}{n^2}\right)$

62. $f(x) = \begin{cases} x^2+1, & x<1 \\ \dfrac{x^2-1}{x-1}, & x>1 \end{cases}$

63. $f(x) = \begin{cases} \alpha x^2+1, & x<2 \\ 5, & x=2 \\ x+3, & 2<x \end{cases}$

64. $\lim\limits_{x \to 1} \dfrac{f(x+2) - f(2x+1)}{x^2-1}$

65. $f(x) = \begin{cases} x^n \sin \frac{1}{x}, & x \neq 0 \\ 0, & x=0 \end{cases}$

66. $\lim\limits_{n \to \infty} (x \operatorname{arctg} nx)$

67. $f(x) = \begin{cases} \dfrac{x^\alpha}{1+e^{-1/x}}, & x \neq 0 \\ 0, & x=0 \end{cases} \quad \alpha \in \mathbb{R}$

2. Nizovi

Predo 2.

68. $\lim_{(n)} \frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}, (p>-1)$

69. $a_n = \frac{1^2+2^2+\ldots+n^2}{n^n}$

70. $a_n = \frac{1+2+3+\ldots+n}{n^2}$

71. $\left\{ \left(1+\frac{1}{n}\right)^{n+1} \right\}_{n=1}^{\infty}$

72. $x_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n} - \ln n, \; n \in \mathbb{N}$

73. $S_n = \frac{1}{2}+\frac{3}{2^2}+\ldots+\frac{2n-1}{2^n}$

74. $\lim_{(n)} \left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n(n+1)} \right)$

75. $\lim_{(n)} \frac{n}{\sqrt[n]{n!}}$

76. $a_n = \frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\ldots+\frac{1}{\sqrt{n^2+n}}$

77. $x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}, \; n \in \mathbb{N}$

78. $x_n = \frac{1}{\ln 2}+\frac{1}{\ln 3}+\ldots+\frac{1}{\ln n}$

79. $\lim_{(n)} \left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{kn} \right)$

80. $\lim_{(n)} \left[\frac{1}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3} \right]$

81. $\lim_{(n)} \left[\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\ldots+(-1)^{n-1}\frac{n}{n} \right]$

82. $\lim_{(n)} \left(\frac{1}{0!}+\frac{1}{1!}+\ldots+\frac{1}{n!} \right)$

83. $a_n = 1+\frac{2}{3}+\frac{3}{3^2}+\ldots+\frac{n+1}{3^n}$

84. $a_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ldots+\frac{1}{\sqrt{n}} - 2\sqrt{n}$

85. $\lim_{(n)} \sqrt[n]{a}$

86. $\lim_{(n)} \frac{x^n}{n!}$

87. $\lim_{(n)} \left\{ \frac{a_1}{a_0 S_1}+\frac{a_2}{S_1 S_2}+\ldots+\frac{a_n}{S_{n-1} S_n} \right\}$

88. $\lim_{(n)} \frac{a_1+a_2+\ldots+a_n}{n}$

89. $a_n = \left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\cdot\ldots\cdot\left(1+\frac{1}{n^3}\right)$

90. $\lim_{(n)} \left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\ldots+\frac{(n-1)^2}{n^3} \right]$

91. $a_n = \left(1+\frac{\alpha}{2^2}\right)\left(1+\frac{\alpha}{3^3}\right)\cdot\ldots\cdot\left(1+\frac{\alpha}{n^n}\right)$

92. Broj e

93. $\lim_{(n)} \sqrt[n]{a_n} = \lim_{(n)} \frac{a_{n+1}}{a_n}$

94. $a_n = \frac{\binom{n}{0}\frac{1}{1}+\binom{n}{1}\frac{1}{2}+\ldots+\binom{n}{n}\frac{1}{n+1}}{2^n}$

95. $\lim_{(n)} \sqrt[n]{a_1 a_2 \cdot\ldots\cdot a_n} = \lim_{(n)} a_n$

96. $\lim_{(n)} \frac{1+\sqrt{2}+\sqrt[3]{3}+\ldots+\sqrt[n]{n}}{n}$

97. $\lim_{(n)} \sqrt[n]{\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2\cdot\ldots\cdot\left(1+\frac{1}{n}\right)^n}$

98. $\left\{ \frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n} \right\}$

99. $a_0 = -\ln 4, \; a_n = \ln \frac{(n+1)(n+3)}{n(n+4)}$

100. $a_n = \frac{3^n+(-2)^n}{3^{n+1}+(-2)^{n+1}} + \left(\frac{n+1}{n-1}\right)^n + (1+\alpha)\sin\frac{n\pi}{2}.$

101. $\lim_{(n)} \left(1+\frac{1}{n}\right)^n$

102. $a_n = \frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdot\ldots\cdot\frac{n^3-1}{n^3+1}$

103. $a_n=\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}$

104. $\lim_{(n)} \frac{a^n}{n!}$

105. $\lim_{(n)} \sqrt[n]{a}$

106. $\lim_{(n)} \frac{n}{3^n}$

107. $\lim_{(n)} \frac{1\cdot2+2\cdot3+\ldots+n(n+1)}{n^3}$

108. $\lim_{(n)} a^n$

109. $\lim_{(n)} \frac{a_0n^k+a_1n^{k-1}+a_2n^{k-2}+\ldots+a_k}{b_0n^h+b_1n^{h-1}+\ldots+b_h}$

110. $\lim_{(n)}\left[\frac{1+3+5+\ldots+(2n-1)}{n+1}-\frac{2n+1}{2}\right]$

111. $\lim_{(n)} n^k a^n$

112. $\lim_{(n)} \left(1-\frac{1}{3}\right)\left(1-\frac{1}{6}\right)\cdot\ldots\cdot\left(1-\frac{1}{\frac{n(n+1)}{2}}\right)$

113. $\lim_{(n)} \left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdot\ldots\cdot\left(1-\frac{1}{n^2}\right)$

3. Redovi

114. $\lim_{n\to-\infty}\left(1+\frac{1}{n}\right)^n$

115. $\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)$

116. $\lim_{x\to0_+}(1+x)^{\frac{1}{x}}$

117. $\lim_{x\to0}\frac{\ln(1+x)}{x}$

118. $\lim_{x\to0}\frac{e^x-1}{x}$

119. $\lim_{x\to0}\frac{\sin x}{x}$

120. $\lim_{x\to+\infty}\frac{x^\alpha}{e^{3x}}$

121. $\lim_{x\to0}\frac{x-\sin x}{x^3}$

122. $\lim_{x\to0}\frac{1-\cos x}{x^2}$

123. $\lim_{x\to0}\frac{\sqrt[7]{1+x\sin x}-1}{\ln^2(1+\operatorname{tg}x)}$

124. $\lim_{n\to+\infty}\frac{\ln^2 n}{n}$

125. $S_n=\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)$

126. $\lim_{x\to0}\frac{\operatorname{sh}x}{x}$

127. $\sum_{1}^{\infty}\sin n\alpha,\ 0<\alpha<\pi$

128. $\lim_{(n)}\frac{a^n}{n!}$

129. $\sum_{n=1}^{\infty}\frac{\cos nx-\cos(n+1)x}{n},\ x\in\mathbb{R}$

130. $\sum_{1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n}}$

131. $\sum_{n=3}^{\infty}\frac{1}{(\ln\ln n)^{\ln n}}$

132. $\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{n}{e}\right)^n$

133. $\sum_{1}^{\infty}\frac{1}{(\ln n)^{\ln\ln n}}$

134. $\sum_{1}^{\infty}\frac{(2n-1)!!}{2n!!}\cdot\frac{1}{2^{n+1}}$

135. $\sum_{n=1}^{\infty} (\sqrt{2}-\sqrt[3]{2})(\sqrt{2}-\sqrt[5]{2})\cdots(\sqrt{2}-\sqrt[2n+1]{2})$

136. $\sum_{n=1}^{\infty} \left(\frac{n-2}{n+2}\right)^{n(n+1)}$

137. $\sum_{1}^{\infty} \frac{n^k}{a^n+b^n}$

138. $\sum_{1}^{\infty} \frac{a+(-1)^n b}{a^n}$

139. $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(a+\sqrt{1})(a+\sqrt{2})\cdots(a+\sqrt{n})}$

140. $\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{2n!!}\right)^{p}$

141. $\sum_{n=1}^{\infty} 5^{\sqrt[3]{n}}$

142. $\sum_{n=1}^{\infty} \left(e^{\frac{1}{n+1}}-1\right)^{p}, \ p\in\mathbb{R}$

143. $\sum_{1}^{\infty} \frac{1}{n^\alpha} \ln\left(n\sin\frac{1}{n}\right)$

144. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}+\ln\left(1+\frac{1}{n}\right)$

145. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}-\ln\left(1+\frac{1}{n}\right)$

146. $\sum_{n=1}^{\infty} \left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$

147. $\sum_{1}^{\infty} \frac{1}{\sqrt{(2n-1)(2n+1)}}$

148. $\left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)\cdot\cos x^n$

149. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha}, \ \alpha\in\mathbb{R}$

150. $\sum_{n=1}^{\infty} (-1)^n\cdot\frac{1}{\ln n}$

151. $\sum_{n=2}^{\infty} \frac{1}{\ln^2 n} \cos\frac{\pi n^2}{n+1}$

152. $\sum_{n=1}^{\infty} (-1)^n\left(\frac{(2n-1)!!}{2n!!}\right)^{\alpha}, \ \alpha\in\mathbb{R}$

153. $\sum_{1}^{\infty} \frac{(-1)^n}{\left(\sqrt{n}+(-1)^{n-1}\right)^p}, \ p\in\mathbb{R}$

154. $\sum_{1}^{\infty} \ln\left(1+\frac{\sqrt[n]{n^2+1}\cdot(-1)^n}{n^\alpha}\right)$

155. $\sum_{n=1}^{\infty} \left(\frac{1}{n}-\ln\left(1+\frac{1}{n}\right)\right)^{\alpha}, \ \alpha\in\mathbb{R}$

156. $\sqrt{2}+\sum_{1}^{\infty}\sqrt{2-\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}}$

157. $\sum_{n=1}^{\infty} (-1)^{n-1}\left[\sqrt{2-\left(\sqrt{2+\sqrt{2+\dots+\sqrt{2}}}\right)^{\alpha}}\right]^{\delta}$

158. $\sum_{1}^{\infty} \frac{\ln^{100} n}{n}\cdot\sin\frac{n\pi}{4}$

159. $\sum_{1}^{\infty} \sin\pi\sqrt{n^2+k^2}, \ k\in\mathbb{N}$

160. $\sum_{n=1}^{\infty} \left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$

161. $a_n=\log_b{}^n\left(1+\frac{\sqrt[n]{a}}{n}\right)$

162. $\sum_{n=2}^{\infty} (-1)^n\frac{1}{n^\alpha\ln^2 n}$

163. $a_n=\left(\cos\frac{1}{\sqrt{n}}\right)^{n^2}$

164. $\sum_{n=1}^{\infty} \arcsin\frac{1}{\sqrt{n}}\cdot\cos\frac{1}{n^2}(-1)^n$

165. $\sum_{1}^{\infty} \frac{(-1)\sqrt{[n]}}{n}$

4. Funkcije

166. $f(x)=2\sin x+\cos 2x$

167. $f(x)=2x-3\sqrt[3]{x^2}$

168. $f(x)=\frac{x^4+1}{x^2}$

169. $y=f(x)$, $(y-x)^2=x^3$

170. $g(x)=f^2(x)=x^2(8-x^2)$

171. $f(x)=(3-x^2)e^x$

172. $y=(x-a)e^{\frac{1}{x}}$

173. $f(x)=\operatorname{arctg}\frac{x+1}{x-1}$

174. $x^4-2x^2=y^3(x-1)$

175. $f(x)=x^3-6x^2+9x-4$

176. $y=\dfrac{x^3-x^2-2x+2}{x}$

177. $f(x)=x^6-3x^4+3x^2$

178. $f(x)=\sqrt[3]{x^2(1-x)^2}$

179. $y=\dfrac{x^3}{3-x^2}$

180. $y=\dfrac{x^4}{x^3-1}$

181. $y=\dfrac{x-2}{\sqrt{x^2+1}}$

182. $f(x)=1-x+\sqrt{\dfrac{x^3}{x+3}}$

183. $y=\dfrac{1}{x(1-\ln x)}$

184. $\lambda x^2-\ln x-1=0$

185. $f(x)=x, \frac{1}{x}, x^2, e^x$

$f(x)=1-xe^{-\frac{2}{x}}$

186. $y=\dfrac{x^2+2}{x}$

187. $y=\sqrt[3]{x^3-x^2}$

188. $y=\dfrac{(x-1)e^{1/x}}{3x-2}$

189. $y=\operatorname{arctg}e^x-x+\frac{1}{2}\ln(1+e^{2x})$

190. $f(x)=\alpha\ln x+x^2+\beta x+\gamma$

5. Integrali

191. $\displaystyle\int\frac{dx}{4x^2+4x+5}$

192. $\displaystyle\int\frac{dx}{\sqrt{1+3x+7x^2}}$

193. $\displaystyle\int\sqrt{3x^2+10x+9}\cdot dx$

194. $\displaystyle\int\frac{4x-3}{5-7x}\,dx$

195. $\displaystyle\int\frac{x^3-2x^2+4}{x^3(x-2)^2}\,dx$

196. $\displaystyle\int\frac{x^3+4x^2-2x+1}{x^4+x}\,dx$

197. $\displaystyle\int\frac{dx}{(x^2+a^2)^2}$

198. $\displaystyle\int\frac{dx}{(x+1)\sqrt{1-x}}$

199. $\displaystyle\int\frac{1}{(1-x)^2}\sqrt{\frac{1-x}{1+x}}\,dx$

200. $\displaystyle\int\frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})}$

201. $\displaystyle\int\frac{dx}{x-\sqrt{x^2-x+1}}$

202. $\int \frac{dx}{3x+\sqrt{5+8x-4x^2}}$

203. $\int \frac{dx}{(x^2+16)\sqrt{9-x^2}}$

204. $\int \frac{x^2-x+1}{(x^2+1)\sqrt{x^2+1}}$

205. $\int \frac{3x^3-8x+5}{\sqrt{x^2-4x-7}}\,dx$

206. $\int (2x-5)\sqrt{2+3x-x^2}\,dx$

207. $\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

208. $\int \frac{(3x-5)dx}{(x-1)^2\sqrt{x^2+4}}$

209. $\int x^5\sqrt[3]{(1+x^3)^2}\,dx$

210. $\int \frac{dx}{x^2\sqrt[3]{(1+x^3)^5}}$

211. $\int x^5\cdot\sqrt[7]{2-3x^6}\,dx$

212. $\int \frac{dx}{\sin x}$

213. $\int \frac{dx}{\cos x}$

214. $\int \frac{2-\sin x}{2+\cos x}\,dx$

215. $\int \frac{dx}{4-3\cos^2 x+5\sin^2 x}$

216. $\int \frac{\sin^3 x}{1+\cos^3 x}\,dx$

217. $\int \frac{\cos^3 x\,dx}{4\sin^2 x-1}$

218. $\int \frac{dx}{\operatorname{sh} x}$

219. $\int \frac{dx}{\operatorname{ch} x}$

220. $\int \frac{dx}{2\operatorname{sh} x+3\operatorname{ch} x}$

221. $\int \sin\frac{x}{4}\cdot\cos\frac{x}{3}\,dx$

222. $\int \cos(ax+b)\cdot\cos(ax-b)\,dx$

223. $\int \sin x\cdot\sin 2x\cdot\sin 3x\,dx$

224. $\int \sin^2 x\,dx$

225. $\int \sin^4 x\,dx$

226. $\int \operatorname{tg}^5 x\,dx$

227. $\int \operatorname{ctg}^7 x\,dx$

228. $\int \frac{dx}{\sin^3 x}$

229. $\int \frac{dx}{\sin^4 x}$

230. $\int \frac{dx}{\cos^5 x}$

231. $\int \sin^2 x\cdot\cos^3 x\,dx$

232. $\int \frac{\cos^3 x}{\sin^4 x}\,dx$

233. $\int \frac{\sin^4 x}{\cos^2 x}\,dx$

234. $\int \sin^5 x \cdot \cos^5 x \, dx$

235. $\int \sin^2 x \cos^2 x \, dx$

236. $\int \frac{dx}{\sin^4 x \cos^4 x}$

237. $\int \sin^2 x \cdot \cos^4 x \, dx$

238. $\int \frac{dx}{\sin^2 x \cdot \cos^4 x}$

239. $\int \frac{dx}{\sin x \cdot \cos^3 x}$

240. $\int \frac{dx}{\sin x \cdot \sin 2x}$

241. $\int \frac{\sin^3 x}{\sqrt[3]{\cos^4 x}} \, dx$

242. $\int \sqrt[5]{\sin^3 x} \cdot \cos^5 x \, dx$

243. $\int_0^{2\pi} |\sin^3 x| \, dx$

244. $\int_{-2}^{-1} \frac{dx}{x^2+4x+5}$

245. $\int_{-2}^{-1} \frac{dx}{(11+5x)^3}$

246. $\int_{\ln 2}^{\ln 3} \frac{dx}{e^x - e^{-x}}$

247. $\int_0^{\pi/2a} (x+3) \sin ax \, dx$

248. $\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} \, dx$

249. $\int_0^{\pi} \frac{dx}{1+\alpha \cos x}$

250. $\int_{-1}^{1} \frac{dx}{x^2 - 2x\cos\alpha + 1}$

251. $\int_0^2 |1-x| \, dx$

252. $\int_0^{\pi/4} \frac{\operatorname{sgn}(ab) \, dx}{a^2 \sin^2 x + b^2 \cos^2 x}$

253. $\int_{1/e}^{e} |\ln x| \, dx$

254. $\int_0^1 (1-x^2)^n \, dx$

255. $\int_{-a}^{a} x^2 \sin x \, dx$

256. $\int_{-1}^{1} \ln \frac{1+\sin x}{1-\sin x} \, dx$

257. $\int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} \, dx$

258. $\int_0^{4\pi} \sqrt{1-\cos x} \cdot dx$

259. $\int_0^{\pi/2} \frac{\sin t}{\sin t + \cos t} \, dt$

260. $\int_1^2 \frac{dx}{1+\sqrt{2x-x^2}}$

261. $\int_0^{\pi} \frac{\sin x}{\sqrt{1-2t\cos x + t^2}} \, dx$

262. $\int_{-\frac{\pi}{2}}^{\pi/2} |\sin x|\sqrt{\cos x}\, dx$

263. $\int_{0}^{\pi/2} \sin x\sqrt{\cos x}\, dx$

264. $\int_{-1}^{4} \frac{[x]}{1+|x|}\, dx$

265. $\int_{0}^{\pi} x\,\mathrm{sgn}(\cos x)\, dx$

266. $\int_{0}^{2} [e^x]\, dx$

267. $\int_{-1}^{1} \frac{dx}{1+x^2}$

268. $\int_{-2}^{2} \frac{dx}{\sqrt{|1-x^2|}}$

269. $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}$

6. Funkcije više promjenjivih

270. $f(x,y) = \begin{cases} x^2y(x^4+y^2)^{-1}, & (x,y) \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$

271. $z = \sqrt[3]{x^3+y^3}$

272. $z = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = 0 \end{cases}$

273. $f(x,y) = \begin{cases} \frac{x^2y}{2(x^4+y^2)} & \text{za } x \neq 0 \text{ i } y \neq 0 \\ 0, & \text{za } x = y = 0 \end{cases}$

274. $f(x,y) = \sqrt[3]{x^3+y^3}$

275. $f(x,y) = \begin{cases} e^{\frac{-1}{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x = y = 0 \end{cases}$

276. $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

277. $z = 6-4x-3y$

$x^2+y^2 = 1$

Limes

Tehnika rješavanja limesa

1. $\lim_{x\to 0} \frac{\sin x}{x} \overset{\frac{0}{0}}{=} 1$

2. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

3. $\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e$

4. $\lim_{x\to +\infty} \frac{x^\alpha}{e^{\delta x}} = 0, \quad \delta>0, \forall\alpha$

5. $\lim_{n\to\infty} \left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \ln n\right) = \gamma = 0{,}572\dots$

6. $\lim_{n\to\infty} \sqrt[n]{n^k} = 1, \quad \forall k\in R$

7. $\lim_{n\to\infty} \sqrt[n]{(\log_a n)^k} = 1, \quad \forall k\in R, \; 0<a\neq 1$

8. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$

9. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

10. $\lim_{x\to 0} \frac{\ln(1+f(x))}{f(x)} = 1$

11. $e^{\ln e} = 1$

12. $e^{\ln(f(x))} = f(x)$

(Z) $\lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{n+3} = L$

$\lim_{n\to\infty} \left(\frac{n+2}{n+1}\right)^{n+3} \overset{1^\infty}{=} \left| \text{zahtijeva reg. transformaciju} \right| = \lim_{n\to\infty} \left(\frac{n+1+1}{n+1}\right)^{n+1+2} =$

$= \lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1+2} = \lim_{n\to\infty} \left[\left(1+\frac{1}{n+1}\right)^{n+1} \cdot \left(1+\frac{1}{n+1}\right)^2\right] =$

$= \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^{n+1}}_{\to e} \cdot \underbrace{\lim_{n\to\infty} \left(1+\frac{1}{n+1}\right)^2}_{\to 1^2} = e\cdot 1^2 = \underline{e}$

(Z) $\lim_{x\to 0}\left(\frac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}} = L$

$\lim_{x\to 0}\left(\frac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}} \overset{1^\infty}{=} \left| Z.R.T. \right| = L / \cdot \ln \Rightarrow$

$\ln L = \lim_{x\to 0} \ln\left(\frac{2^x+3^x+4^x}{3}\right)^{\frac{1}{x}} = \lim_{x\to 0} \frac{\ln\left(\frac{2^x+3^x+4^x}{3}\right)}{x} = \left| \begin{array}{l} \ln u \sim u-1 \\ u\to 1 \end{array} \right| =$

$= \text{ili} \left| \begin{array}{l} \text{ŠTIMAMO:} \\ \frac{\ln(1+x)}{x} \to 1, \text{ kad } x\to 0 \end{array} \right| = \left| \begin{array}{l} \text{kod nas je} \\ u = \left(\frac{2^x+3^x+4^x}{3}\right) \to 1 \text{ kod } x\to 0 \text{ pa je} \end{array} \right| =$

$= \lim_{x\to 0} \frac{\frac{2^x+3^x+4^x}{3}-1}{x} = \lim_{x\to 0}\left(\frac{2^x+3^x+4^x-3}{3x}\right) \overset{\frac{0}{0}}{=} \left| Z.R.T \right| =$

$= \left| \begin{array}{l} \text{ŠTIMAMO NA FAZON:} \\ \lim_{x\to 0} \frac{e^x-1}{x} = 1 \end{array} \right| = \lim_{x\to 0}\left(\frac{2^x-1+3^x-1+4^x-1-3+3}{3x}\right) =$

$= \lim_{x\to 0} \frac{1}{3}\left(\frac{e^{\ln 2^x}-1+e^{\ln 3^x}-1+e^{\ln 4^x}-1}{x}\right) = \lim_{x\to 0}\frac{1}{3}\cdot\left[\frac{e^{\ln 2^x}-1}{x}+\frac{e^{\ln 3^x}-1}{x}+\right.$

$\left.+\frac{e^{\ln 4^x}-1}{x}\right] = \frac{1}{3}\lim_{x\to 0}\left[\frac{e^{\ln 2^x}-1}{x\cdot\ln 2}\cdot\ln 2+\frac{e^{\ln 3^x}-1}{x\cdot\ln 3}\cdot\ln 3+\frac{e^{\ln 4^x}-1}{x\cdot\ln 4}\cdot\ln 4\right] =$

$= \left| \frac{e^{x\ln 2}-1}{x\cdot\ln 2} \to 1, \text{ kad } x\to 0 \right| = \frac{1}{3}(\ln 2+\ln 3+\ln 4).$

$\ln L = \frac{1}{3}(\ln 2+\ln 3+\ln 4) = \frac{1}{3}(\ln(2\cdot 3\cdot 4)) = \frac{1}{3}\ln 24 =$

$= \ln 24^{1/3} \Rightarrow \qquad \underline{L = \sqrt[3]{24}}.$

* Upotrebljene su sljedeće transformacije:

1. $\lim_{u\to 1} \ln u \sim u-1$

2. $\lim_{u\to 0} \frac{\ln(1+u)}{u} \to 1$

3. $e^{\ln e} = 1$

4. $e^{\ln(f(x))} = f(x)$

5. $\lim_{x\to 0} \frac{e^x-1}{x} = 1$

(Z) $\lim_{x\to 0}\left(\frac{\operatorname{arctg} x}{x}\right)^{1/x^2} = L$

$\lim_{x\to 0}\left(\frac{\operatorname{arctg} x}{x}\right)^{\frac{1}{x^2}} \overset{1^\infty}{=} \left| \begin{array}{l} \text{zahtijeva} \\ \text{regularnu} \\ \text{smjenu} \end{array} \right| = L \;/\circ \ln \Rightarrow$

$$\Rightarrow \ln L = \lim_{x\to 0} \frac{1}{x^2}\cdot \ln\left(\frac{\operatorname{arctg} x}{x}\right) = \left| \begin{matrix} \ln x \sim x-1 \\ x\to 1 \end{matrix} \right| =$$

$$= \lim_{x\to 0} \frac{1}{x^2}\left(\frac{\operatorname{arctg} x}{x} - 1\right) = \lim_{x\to 0}\left(\frac{\operatorname{arctg} x - x}{x^3}\right) \overset{\frac{0}{0}}{=} \left| \text{Z.R.T.} \right| =$$

$$= \left| \left.\begin{matrix} \operatorname{arctg} x = u \\ u \to 0 \end{matrix}\right\} \Rightarrow x = \operatorname{tg} u \right| = \lim_{u\to 0}\left(\frac{u - \operatorname{tg} u}{(\operatorname{tg} u)^3}\right) \overset{\frac{0}{0}}{=} \left| \begin{matrix} \text{ŠTIMUNG NA:} \\ \left(\frac{\operatorname{tg} u}{u}\right)^3 \to 1,\ u\to 0 \end{matrix} \right| =$$

$$= \lim_{u\to 0} \frac{u - \operatorname{tg} u}{\underbrace{\frac{\operatorname{tg}^3 u}{u^3}}_{\to 1}\cdot u^3} = \lim_{u\to 0} \frac{u - \operatorname{tg} u}{u^3} \overset{\frac{0}{0}}{=} \lim_{u\to 0} \frac{u - \frac{\sin u}{\cos u}}{u^3} =$$

$$= \lim_{u\to 0} \frac{u\cos u - \sin u}{u^3 \cdot \underbrace{\cos u}_{\to 1}} = \lim_{u\to 0} \frac{u\cos u - \sin u}{u^3} \overset{\frac{0}{0}}{=} \lim_{u\to 0} \frac{u - u + u\cos u - \sin u}{u^3} =$$

$$= \lim_{u\to 0} \frac{u - \sin u}{u^3} + \lim_{u\to 0} \frac{u\cos u - u}{u^3} = \left| \begin{matrix} \text{NAJPRIJE ĆEMO RIJEŠITI} \\ \text{JEDAN PA DRUGI} \\ \text{LIMES} \end{matrix} \right. \Rightarrow$$

$$* \lim_{u\to 0} \frac{u\cos u - u}{u^3} = -\lim_{u\to 0} \frac{(1-\cos u)(1+\cos u)}{u^2\cdot(1+\cos u)} = -\lim_{u\to 0} \frac{1-\cos^2 u}{u^2}\left(\frac{1}{1+\cos u}\right) =$$

$$= \left| \begin{matrix} \text{ŠTIMALI SMO NA OBLIK} \\ \frac{\sin x}{x} \end{matrix} \right| = -\lim_{u\to 0} \underbrace{\frac{\sin^2 u}{u^2}}_{\to 1} \underbrace{\left(\frac{1}{1+\cos u}\right)}_{\to \frac{1}{2}} = -\frac{1}{2}.$$

$$* \lim_{u\to 0} \frac{u - \sin u}{u^3} \overset{\frac{0}{0}}{=} \left| \begin{matrix} u = 2x \\ x \to 0 \end{matrix} \right| = \lim_{x\to 0} \frac{2x - \sin 2x}{8x^3} = \left| \sin 2x = 2\sin x\cos x \right| =$$

$$= \lim_{x\to 0} \frac{2x - 2\sin x\cos x}{8x^3} = \frac{1}{4}\lim_{x\to 0}\left(\frac{x - \sin x\cdot\cos x}{x^3}\right) = \frac{1}{4}\lim_{x\to 0}\left(\frac{x + \sin x - \sin x - \sin x\cos x}{x^3}\right) =$$

$$= \underbrace{\left(\lim_{x\to 0} \frac{x - \sin x}{x^3}\right)}_{L_0}\cdot\frac{1}{4} + \frac{1}{4}\lim_{x\to 0} \frac{\sin x(1-\cos x)}{x^3} =$$

$$= \frac{1}{4}L_0 + \frac{1}{4}\lim_{x\to 0} \frac{\sin x(1-\cos^2 x)}{x^3\underbrace{(1+\cos x)}_{\to 2}} = \frac{1}{4}L_0 + \frac{1}{4}\cdot\frac{1}{2} = \frac{1}{4}L_0 + \frac{1}{8},$$

$$L_0 = \frac{1}{4}L_0 + \frac{1}{8} \Rightarrow L_0 = \frac{1}{6}$$

$$\ln L = \underbrace{\frac{1}{4}L_0 + \frac{1}{8}}_{\frac{1}{6}} - \frac{1}{2} = -\frac{1}{3} \Rightarrow \ln L = -\frac{1}{3} \Rightarrow L = e^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{e}}$$

$$L = \frac{1}{\sqrt[3]{e}}.$$

* UPOTREBLJENE TRANSFORMACIJE:

1. $\lim_{x\to 0} \frac{\sin x}{x} = 1$
2. $\lim_{u\to 0} \cos u = 1$
3. $\ln x \sim x-1,\ x\to 1$

* UPOTREBLJENE SMJENE:

1. $\sin 2x = 2\sin x\cos x$
2. $u = \operatorname{arctg} x$

(Z) $$\lim_{x\to 0}\frac{e^x-e^{\operatorname{tg}x}}{x-\operatorname{tg}x} \overset{\frac{0}{0}}{=} |Z.R.T.| = \left|\begin{matrix}\text{ŠTIMAMO NA:}\\ \lim_{x\to 0}\frac{e^x-1}{x}=1\end{matrix}\right| =$$

$$= \lim_{x\to 0}\frac{e^x\left(1-\frac{e^{\operatorname{tg}x}}{e^x}\right)}{x-\operatorname{tg}x} = \lim_{x\to 0}\frac{-e^x(e^{\operatorname{tg}x}\cdot e^{-x}-1)}{-(\operatorname{tg}x-x)} = \lim_{x\to 0}\frac{e^x(e^{\operatorname{tg}x-x}-1)}{\operatorname{tg}x-x} =$$

$$= \lim_{x\to 0}e^x\cdot\underbrace{\lim_{x\to 0}\frac{e^{\operatorname{tg}x-x}-1}{\operatorname{tg}x-x}}_{\to 1} = \lim_{x\to 0}e^x = \underline{1}.$$

(Z) $$\lim_{x\to 0}\frac{1-\sqrt{\cos x}\cdot\sqrt[7]{\cos 5x}}{x^2} = L$$

$$\lim_{x\to 0}\frac{1-\sqrt{\cos x}\cdot\sqrt[7]{\cos 5x}}{x^2} \overset{\frac{0}{0}}{=} |Z.R.T| = |e^{\ln f(x)} = f(x)| = \lim_{x\to 0}\frac{1-(\cos x)^{\frac{1}{2}}\cdot(\cos 5x)^{\frac{1}{7}}}{x^2} =$$

$$= \lim_{x\to 0}\frac{1-e^{\ln(\cos x)^{\frac{1}{2}}}\cdot e^{\ln(\cos 5x)^{1/7}}}{x^2} = -\lim_{x\to 0}\left(\frac{e^{\frac{1}{2}\ln\cos x+\frac{1}{7}\ln\cos 5x}-1}{x^2}\right) \overset{\frac{0}{0}}{=} |Z.R.T.| =$$

$$= -\lim_{x\to 0}\frac{e^{\frac{1}{2}\ln\cos x+\frac{1}{7}\ln\cos 5x}-1}{x^2} = \left|\begin{matrix}\text{ŠTIMAMO NA:}\\ \lim_{x\to 0}\frac{e^x-1}{x}=1\end{matrix}\right| =$$

$$= -\lim_{x\to 0}\underbrace{\frac{e^{\frac{1}{2}\ln\cos x+\frac{1}{7}\ln\cos 5x}-1}{\frac{1}{2}\ln\cos x+\frac{1}{7}\ln\cos 5x}}_{\to 1}\cdot\frac{\frac{1}{2}\ln\cos x+\frac{1}{7}\ln\cos 5x}{x^2} =$$

$$= -\lim_{x\to 0}\frac{\frac{1}{2}\ln\cos x+\frac{1}{7}\ln\cos 5x}{x^2} = \left|\begin{matrix}\ln u \sim u-1\\ u\to 1\end{matrix}\right| = -\lim_{x\to 0}\frac{\frac{1}{2}(\cos x-1)+\frac{1}{7}(\cos 5x-1)}{x^2} =$$

$$= +\lim_{x\to 0}\frac{\frac{1}{2}(1-\cos^2 x)}{x^2(1+\cos x)} + \lim_{x\to 0}\frac{\frac{1}{7}(1-\cos^2 5x)\cdot 25}{(5x)^2(1+\cos 5x)} = \lim_{x\to 0}\underbrace{\frac{\sin^2 x}{x^2}}_{\to 1}\cdot\frac{1}{\underbrace{2(1+\cos x)}_{\to 2}} +$$

$$+ \lim_{x\to 0}\underbrace{\frac{\sin^2 5x}{(5x)^2}}_{\to 1}\cdot\frac{25}{\underbrace{7\cdot(1+\cos 5x)}_{\to 2}} = \frac{1}{4}+\frac{25}{14} = \underline{\frac{57}{28}}.$$

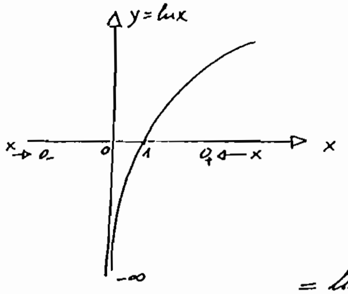
* Zadatak je pojednostavljen kad se došlo do dijela: $-\lim_{x\to 0}\frac{\frac{1}{2}\ln\cos x+\frac{1}{7}\cos 5x}{x^2}$ jer se tada koristila smjena:

$\ln u \sim u-1$ ...

Moglo se točiti na formulu: $\lim_{u\to 0}\frac{\ln u}{u-1}=1$ ali bi tada račun bio duži!

(Z) $\lim_{x\to 0_+} x^2\cdot \ln^3 x = L$

$$\lim_{x\to 0_+} x^2\cdot \ln^3 x \overset{0\cdot\infty}{=} \left|\begin{matrix}z.\ k.\ T.\\ ili\ S.\end{matrix}\right| = \left|\begin{matrix}\ln x = -u, & u\to+\infty\\ x = e^{-u} = \frac{1}{e^u} & \end{matrix}\right| =$$



Sa slike je vidljivo:

$$\boxed{\ln 0_+ = -\infty}$$

$$= \lim_{t\to+\infty} e^{-2t}(-t)^3 = \lim_{t\to+\infty}\frac{(-t)^3}{e^{2t}} =$$

$$= -\lim_{t\to+\infty}\frac{t^3}{e^{2t}} = \left|\begin{matrix}x = t\\ x\to+\infty\end{matrix}\right| = -\lim_{x\to+\infty}\frac{x^3}{e^{2x}} = \left|\begin{matrix}\text{kako je } \lim_{x\to+\infty}\frac{x^\alpha}{e^{\delta x}} = 0\\ \alpha\in\mathbb{R},\ \delta>0\end{matrix}\right| = \underline{0_-}.$$

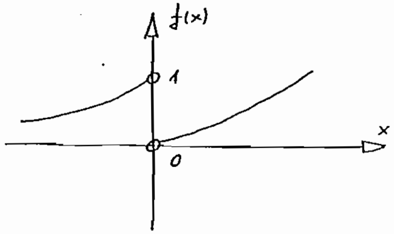
$\Rightarrow$ $\underline{L = 0_-}$.

(Z) Ako je $f(x)=\frac{1}{1+2^{1/x}}$ nađi L_1, L_2, L kad $x\to 0$!

$$L_1 = \lim_{x\to 0_-}\frac{1}{1+2^{1/x}} = \left|\begin{matrix}\frac{1}{0_-} = -\infty\\ 2^{-\infty} = \frac{1}{2^\infty} = \frac{1}{\infty} = 0\end{matrix}\right| = \frac{1}{1+0} = 1.$$

$$L_2 = \lim_{x\to 0_+}\frac{1}{1+2^{1/x}} = \left|\begin{matrix}\frac{1}{0_+} = +\infty\\ 2^{+\infty} = +\infty\end{matrix}\right| = \frac{1}{+\infty} = +0.$$

✳ $|L_1 - L_2| = |1-0| = \underline{1}$. Ovo je tzv. skok f-je, a ujedno je i $L_1 \neq L_2$ pa ne postoji L. To je tačka prekida f-je i to tačka prekida I vrste. Zbog $L_1\neq L_2$ limes divergira u širem smislu (O.Š.S.)!



$f(x)=\frac{1}{1+2^{1/x}}$,

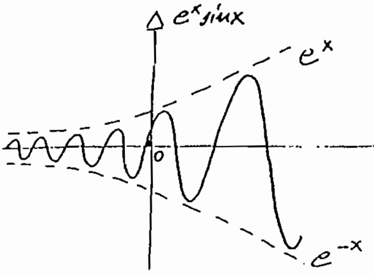
$D(f) = x\in\mathbb{R}\setminus\{0\}$

f-ja

////////////////////

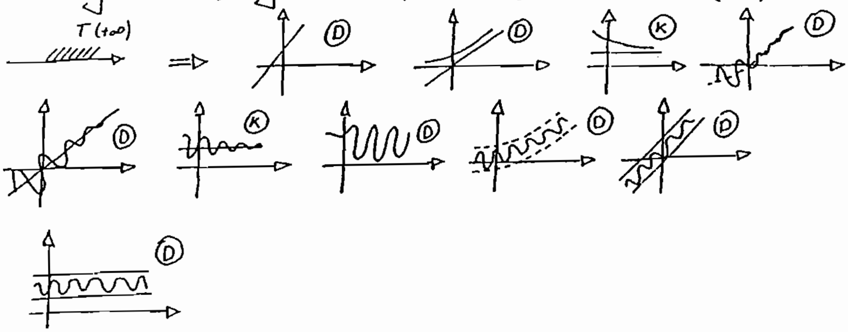
② Ispitajmo limes: $\lim_{x \to +\infty} e^x \sin x = L = \begin{cases} +\infty \\ -\infty \\ 0 \\ \vdots \end{cases}$ zašto!?

$e^x > 0, \forall x$ pa se $e^x \sin x$ ponaša kao $\sin x$.



$\Rightarrow$ da $\lim_{x \to +\infty} e^x \sin x$ divergira u širem smislu (dšs.).

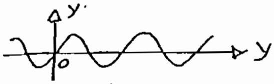
✳ Sljedeći slučajevi su vezani za tačku $T(+\infty)$:

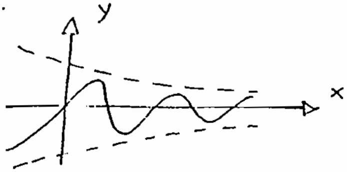


$\lim_{x \to +\infty} e^{-x} \sin x = 0 \Rightarrow$ iz $0 < |e^{-x} \sin x| < e^{-x} \;/\cdot \lim_{x \to +\infty} \Rightarrow$

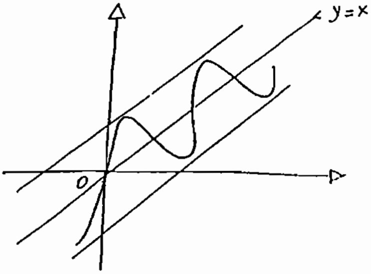
$0 \le \lim_{x \to +\infty} |e^{-x} \sin x| \le \lim_{x \to +\infty} \frac{1}{e^x} = 0 \Rightarrow$

$\lim_{x \to +\infty} |e^{-x} \sin x| = 0 \Leftrightarrow \lim_{x \to +\infty} e^{-x} \sin x = 0$

Sinx - elementarna f-ja  , međutim $\frac{1}{e^x}$ kad $x \to +\infty$ približava se nuli ; kada broj koji je manji od 1 množimo sa sinx dobijemo broj koji je manji od sinx.



(Z) $\lim\limits_{x\to+\infty}(x+\sin x) = L$ (D) u širem smislu



(Z) $y = x\sqrt{\dfrac{x-1}{x+1}}$

* kada nađemo definiciono područje f-je potrebno je izvršiti i analizu tog područja.

* Pravila za definiciono područje:

1. $\dfrac{1}{h(x)}$, $h(x)\neq 0$

2. $\sqrt[2n]{h(x)}$, $h(x)\geq 0$

3. $\log_a(h(x))$, $h(x)>0$

4. $\arcsin(h(x))$, $|h(x)|\leq 1$

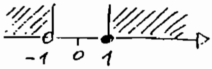
* $D(f)$

1° $\dfrac{x-1}{x+1}\geq 0 \Rightarrow$ tablica $\Rightarrow$

	$-\infty$	-1	1	$+\infty$
$x+1$	$-$	0 $+$	$+$	
$x-1$	$-$	$-$	0 $+$	
	⊕	⊖	⊕	

2° $x+1\neq 0 \Rightarrow x\neq -1$

$D(f)$: $x\in(-\infty,-1)\cup\{0\}\cup[1,\infty)$ tj.



Definiciono područje analiziramo limesom!

$\lim\limits_{x\to-\infty} x\sqrt{\dfrac{x-1}{x+1}} = \lim\limits_{x\to-\infty} x\cdot\sqrt{\dfrac{x+1-1-1}{x+1}} = \lim\limits_{x\to-\infty} x\sqrt{1-\dfrac{2}{x+1}} =$

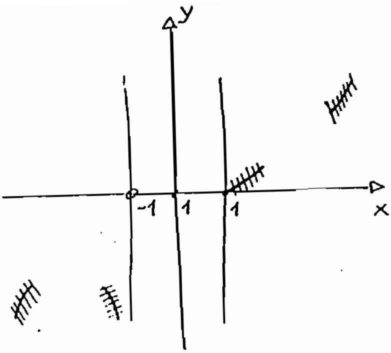
$= \lim\limits_{x\to-\infty}\sqrt{x^2-\dfrac{2x^2}{x+1}} = \lim\limits_{x\to-\infty}\sqrt{1-\dfrac{2}{\frac{1}{x}+\frac{1}{x^2}}} = 1-\infty = \underline{-\infty}.$

$\lim\limits_{x\to+\infty} x\sqrt{\dfrac{x-1}{x+1}} = \underline{+\infty}.$

$y(0)=0$

$y(1)=0$

Crtamo pomoćni dijagram:



Sada bi mogli naći kosu asimptotu!

1) mora biti $\lim_{x\to+\infty} f(x)=+\infty$

ili $\lim_{x\to+\infty} f(x)=-\infty$

2) $y=kx+b$, $x\to+\infty$ (Moguća kosa asimptota)

$k=\frac{y}{x}-\frac{b}{x} \Rightarrow k=\lim_{x\to+\infty}\frac{y}{x}$

* Ako nađemo $k\neq 0$ tražimo b ovako: $b=\lim_{x\to+\infty}(y-kx)$
* Ako je $k=0$ ili ∞ nema kose asimptote
* Ako je b konačno $\Rightarrow$ k.AS. u obliku $y=kx+b$.

1) $y=x\sqrt{\frac{x-1}{x+1}} \xrightarrow{x\to+\infty} +\infty$, zadovoljava

2) $k=\lim_{x\to+\infty}\frac{x\sqrt{\frac{x-1}{x+1}}}{x}=\lim_{x\to+\infty}\sqrt{\frac{x-1}{x+1}}=\lim_{x\to+\infty}\sqrt{1-\frac{2}{x+1}}=1$

$k=1\neq 0$ zadovoljava $\Rightarrow$ traženje parametra b!

$b=\lim_{x\to+\infty}\left(x\sqrt{\frac{x-1}{x+1}}-k\cdot x\right)=\lim_{x\to+\infty}x\left(\frac{\sqrt{x-1}}{\sqrt{x+1}}-1\right)=\lim_{x\to+\infty}x\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x+1}}\right)=$

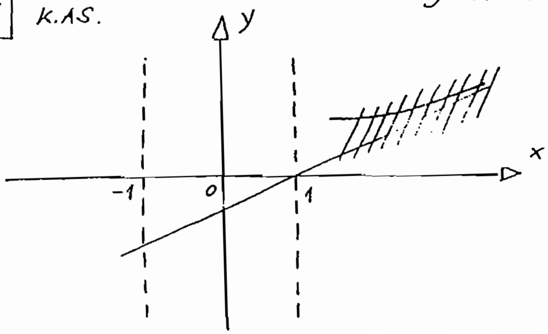
$=\lim_{x\to+\infty}x\left(\frac{(\sqrt{x-1})^2-(\sqrt{x+1})^2}{\sqrt{x+1}(\sqrt{x-1}+\sqrt{x+1})}\right)=\lim_{x\to+\infty}x\cdot\left(\frac{x-1-x-1}{x+1+\sqrt{x+1}\sqrt{x-1}}\right)=$

$=\lim_{x\to+\infty}\frac{-2x}{x+1+\sqrt{x^2-1}}=\lim_{x\to+\infty}\frac{-2}{1+\frac{1}{x}+\sqrt{1-1/x^2}}=-\frac{2}{2}=\underline{-1}.$

$b=-1 \Rightarrow$ kosa asimptota je:

$y=1\cdot x-1=x-1$

$\boxed{y=x-1}$ k.AS.



Z a) $$\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}=\left|\lim_{x\to\infty}\frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}}\quad \text{ovo NIKADA jer } x\to 0, \text{ pa je najbolje uvoditi Limes direktno.}\right|=$$

$$=\lim_{x\to 0}\frac{x^2-1}{2x^2-x-1}=\frac{-1}{-1}=\underline{1}.$$

b) $$\lim_{x\to+\infty}\frac{x^2-1}{2x^2-x-1}\overset{\frac{\infty}{\infty}}{=}\lim_{x\to+\infty}\frac{1-\frac{1}{x^2}}{2-\frac{1}{x}-\frac{1}{x^2}}=\underline{\frac{1}{2}}.$$

c) $$\lim_{x\to 1}\frac{x^2-1}{2x^2-x-1}\overset{\frac{0}{0}}{=}\frac{(x-1)(x+1)}{2x^2-x-1}=\left|\begin{array}{l}2x^2-x-1=0\\ x_{1,2}=\frac{1\pm\sqrt{1+8}}{4}=\frac{1\pm 3}{4},\quad x_{1,2}=-\frac{1}{2},1\end{array}\right|=$$

$$=\lim_{\substack{x\to 1\\ x\neq 1}}\frac{(x-1)(x+1)}{2(x+\frac{1}{2})(x-1)}=\lim_{x\to 1}\frac{x+1}{2(x+\frac{1}{2})}=\frac{2}{2+1}=\underline{\frac{2}{3}}.$$

Z $$L=\lim_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2+x^5}=\lim_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2\underbrace{(1+x^3)}_{\to 1}}=$$

$$=\lim_{x\to 0}\frac{(1+x)^5-(1+5x)}{x^2}\overset{\frac{0}{0}}{=}|Z.R.T|=$$

$$=\left|\begin{array}{l}(1+x)^5=\binom{5}{0}+\binom{5}{1}x+\binom{5}{2}x^2+\binom{5}{3}x^3+\binom{5}{4}x^4+\binom{5}{5}x^5=1+5x+10x^2+10x^3+5x^4+x^5\\ (a+b)^n=a^n+\binom{n}{1}a^{n-1}b+\binom{n}{2}a^{n-2}b^2+\ldots+\binom{n}{n-2}a^2b^{n-2}+\binom{n}{n-1}ab^{n-1}+b^n\end{array}\right|=$$

$$=\lim_{x\to 0}\frac{(1+5x)+10x^2+10x^3+5x^4+x^5-(1+5x)}{x^2}=\lim_{x\to 0}(10+10x+5x^2+x^3)=$$

$$=\lim_{x\to 0}10=\underline{10}.$$

Z $$\lim_{x\to+\infty}\frac{(x+1)(x^2+1)\cdot\ldots\cdot(x^n+1)}{\left((nx)^n+1\right)^{\frac{n+1}{2}}}\overset{\frac{\infty}{\infty}}{=}|Z.R.T.|=\left|\text{ispred svake zagrade u brojniku izvlačimo } x \text{ na stepen koji ima u zagradi}\right|=$$

$$=\lim_{x\to+\infty}\frac{x^{1+2+3+\ldots+n}\cdot\overbrace{(1+\frac{1}{x})(1+\frac{1}{x^2})\cdot\ldots\cdot(1+\frac{1}{x^n})}^{\to 1}}{\left((nx)^n+1\right)^{\frac{n+1}{2}}}=\lim_{x\to\infty}\frac{x^{1+2+\ldots+n}\cdot 1}{(x^n\cdot n^n+1)^{\frac{n+1}{2}}}=$$

$$=\left|\begin{array}{l}1+2+3+\ldots+n=S_n=\frac{n}{2}(a_1+a_n)\\ S_n=\frac{n}{2}(1+n)\end{array}\right|=\lim_{x\to+\infty}\frac{x^{\frac{n(n+1)}{2}}}{x^{\frac{n(n+1)}{2}}\Big(n^n+\underbrace{\frac{1}{x^n}}_{\to 0}\Big)^{\frac{n+1}{2}}}=$$

$$=\lim_{x\to+\infty}\frac{1}{\left(n^n+\frac{1}{x^n}\right)^{\frac{n+1}{2}}}=\lim_{x\to+\infty}\frac{1}{n^{\frac{n\cdot(n+1)}{2}}}=\underline{\frac{1}{n^{\frac{n}{2}(n+1)}}}.$$

(Z) $\lim\limits_{x\to -1} \dfrac{x^3-2x-1}{x^5-2x-1} \overset{\frac{0}{0}}{=} L$

-1	1	0	-2	1
	1	-1	-1	0

x^3-2x-1 je djeljivo sa $(x+1)$

$(x^3-2x-1):(x+1) = x^2-x-1$

$\pm x^3 \pm x^2$

$-x^2-2x$

$-x^2-x$

$+ \quad +$

$-x-1$

$-x-1$

$+ \quad +$

$// \quad //$

$(x^5-2x-1):(x+1) = x^4-x^3+x^2-x-1$

$\pm x^5 \pm x^4$

$-x^4-2x$

$-x^4-x^3$

$+ \quad +$

$+x^3-2x$

$+x^3+x^2$

$- \quad -$

$-x^2-2x$

$-x^2-x$

$+ \quad +$

$-x-1$

$-x-1$

$+ \quad +$

$// \quad //$

1) $(x^3-2x-1) = (x+1)(x^2-x-1)$

2) $(x^5-2x-1) = (x+1)(x^4-x^3+x^2-x-1)$

$$\lim_{x\to -1} \frac{(x+1)(x^2-x-1)}{(x+1)(x^4-x^3+x^2-x-1)} = \frac{1+1-1}{1+1+1+1-1} = \underline{\frac{1}{3}}.$$

(Z) $\lim\limits_{x\to +16} \dfrac{\sqrt[4]{x}-2}{\sqrt{x}-4} = L$

$$\lim_{x\to 16} \frac{\sqrt[4]{x}-2}{\sqrt{x}-4} \overset{\frac{0}{0}}{=} |Z.R.T| = \left| \begin{matrix} x=t^4, \\ t\to 2 \end{matrix} \right| = \lim_{t\to 2} \frac{t-2}{t^2-4} = \lim_{t\to 2} \frac{(t-2)}{(t-2)(t+2)} =$$

$$= \lim_{t\to 2} \frac{1}{t+2} = \underline{\frac{1}{4}}.$$

* (1) $\sqrt{a}-\sqrt{b} = \dfrac{a-b}{\sqrt{a}+\sqrt{b}}, \quad a>0, \quad b>0$

(2) $\alpha\sqrt{a}-\beta\sqrt{b} = \dfrac{\alpha^2 a-\beta^2 b}{\alpha\sqrt{a}+\beta\sqrt{b}}, \quad a>0, \ b>0$

(3) $\sqrt[3]{a}-\sqrt[3]{b} = \dfrac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{a\cdot b}+\sqrt[3]{b^2}}$

npr. $\sqrt[3]{x}-2 = \sqrt[3]{x}-\sqrt[3]{8}$

(Z) $\lim\limits_{x\to 8} \dfrac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} = L$

$$\lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} \overset{\frac{0}{0}}{=} |Z.R.T.| = \lim_{x\to 8} \frac{\sqrt{9+2x}-5}{\sqrt[3]{x}-2} \Big/ \cdot \frac{\sqrt{9+2x}+5}{\sqrt{9+2x}+5} \Rightarrow$$

$$= \lim_{x\to 8} \frac{9+2x-25}{(\sqrt{9+2x}+5)(\sqrt[3]{x}-2)} = \left| \frac{1}{\sqrt[3]{x}-2} = \frac{1}{\sqrt[3]{x}-\sqrt[3]{8}} = \frac{\sqrt[3]{x^2}+\sqrt[3]{8x}+\sqrt[3]{8^2}}{x-8} \right| =$$

$$= \lim_{x\to 8} \frac{2x-16}{(\sqrt{9+2x}+5)(x-8)} = \lim_{x\to 8} \frac{(2x-16)\cdot 12}{(\sqrt{9+2x}+5)(x-8)} = \lim_{x\to 8} \frac{2(x-8)\cdot 12}{(x-8)\sqrt{9+2x}+5} =$$

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(Z) $\lim\limits_{x\to 1}(1-x)\,\mathrm{tg}\,\frac{\pi x}{2} \overset{0\cdot\infty}{=} \lim\limits_{x\to 1}(1-x)\frac{\overbrace{\sin \pi x/2}^{\to 1}}{\cos\frac{\pi x}{2}} = \lim\limits_{x\to 1}\frac{(1-x)}{\cos\frac{\pi x}{2}} \overset{\frac{0}{0}}{=} |Z.R.T| =$

$= \lim\limits_{t\to 0} t\cdot\frac{1}{\cos\frac{\pi}{2}(1-t)} = \left|\begin{array}{l}\text{dobijeno iz:}\\ 1-x=t\\ t\to 0,\ x\to 1\end{array}\right| = \lim\limits_{t\to 0}\frac{t}{\cos\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)} \overset{\frac{0}{0}}{=} |Z.R.T.| =$

$= \left|\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta\right| = \lim\limits_{t\to 0}\frac{t}{\underbrace{\cos\frac{\pi}{2}}_{\to 0}\cdot\cos\frac{\pi t}{2} + \underbrace{\sin\frac{\pi}{2}}_{\to 1}\sin\frac{\pi t}{2}} =$

$= \lim\limits_{t\to 0}\frac{t}{0+\sin\frac{\pi t}{2}} = \lim\limits_{t\to 0}\frac{t}{\sin\frac{\pi t}{2}} = \lim\limits_{t\to 0}\left(\frac{\sin\frac{\pi t}{2}}{t}\right)^{-1}\cdot\frac{\pi/2}{\pi/2} =$

$= \lim\limits_{t\to 0}\left(\frac{\sin\frac{\pi t}{2}}{\frac{\pi t}{2}}\right)^{-1}\cdot\frac{1}{\pi/2} = 1^{-1}\cdot\frac{2}{\pi} = \underline{\frac{2}{\pi}}.$

(Z) $\lim\limits_{x\to+\infty}\frac{\ln(2+e^{3x})}{\ln(3+e^{2x})} = \lim\limits_{x\to+\infty}\frac{\ln e^{3x}\left(\overbrace{\frac{2}{e^{3x}}}^{\to 0}+1\right)}{\ln e^{2x}\left(\underbrace{\frac{3}{e^{2x}}}_{\to 0}+1\right)} =$

$= \lim\limits_{x\to+\infty}\frac{3x+\overbrace{\ln 1}^{\to 1}}{2x+\underbrace{\ln 1}_{\to 0}} = \lim\limits_{x\to+\infty}\frac{3x}{2x} = \underline{\frac{3}{2}}.$

(Z) $\lim\limits_{n\to\infty}\cos^n\frac{x}{\sqrt{n}} \overset{1^\infty}{=} L(x);\quad (L(0)=0)\quad /\circ\ln$

$\ln L = \lim\limits_{n\to\infty} n\ln\cos\frac{x}{\sqrt{n}} = \lim\limits_{n\to\infty} n\frac{\ln\cos\frac{x}{\sqrt{n}}}{\underbrace{\left(\cos\frac{x}{\sqrt{n}}+1\right)}_{\to 1}}\cdot\left(\cos\frac{x}{\sqrt{n}}+1\right) =$

$= \left|\begin{array}{l}\text{jer smo uzeli smjenu}\\ \ln\cos\frac{x}{\sqrt{n}} = \left(\cos\frac{x}{\sqrt{n}}-1\right)\end{array}\right| = +\lim\limits_{n\to\infty} n\cdot\ln\cos\frac{x}{\sqrt{n}}\cdot\frac{\left(1+\cos\frac{x}{\sqrt{n}}\right)}{\left(1+\cos\frac{x}{\sqrt{n}}\right)} =$

$= \left|\begin{array}{l}\ln\cos\frac{x}{\sqrt{n}} \sim \cos\frac{x}{\sqrt{n}}-1\\ \cos\frac{x}{\sqrt{n}}\to 1\end{array}\right| = \lim\limits_{n\to\infty} n\cdot\frac{\left(\cos\frac{x}{\sqrt{n}}-1\right)\left(1+\cos\frac{x}{\sqrt{n}}\right)}{\underbrace{1+\cos\frac{x}{\sqrt{n}}}_{\to 2}} =$

$= -\lim\limits_{n\to\infty} n\frac{1}{2}\left(1-\cos\frac{x}{\sqrt{n}}\right)\left(1+\cos\frac{x}{\sqrt{n}}\right) = -\lim\limits_{n\to\infty}\frac{1}{2}n\left(1-\cos^2\frac{x}{\sqrt{n}}\right) =$

$= -\lim\limits_{n\to\infty}\frac{1}{2}n\sin^2\frac{x}{\sqrt{n}} = \left|\begin{array}{l}\sin x\sim x,\\ x\to 0\end{array}\right| = -\lim\limits_{n\to\infty} n\cdot\frac{x^2}{n} = -\frac{x^2}{2}$

$\ln L = -\frac{x^2}{2} \Rightarrow L = e^{-\frac{x^2}{2}} = \underline{\frac{1}{\sqrt{e^{x^2}}}}.$

(Z) $\lim\limits_{x\to 0_+} \sqrt[x]{\cos\sqrt{x}} = L$

$\lim\limits_{x\to 0_+} \sqrt[x]{\cos\sqrt{x}} = L \;/\cdot \ln \Rightarrow \lim\limits_{x\to 0_+} \frac{1}{x}\ln\cos\sqrt{x} = |2\,K.T| =$

$= \lim\limits_{x\to 0_+} \frac{1}{x}(\cos\sqrt{x}-1) = -\lim\limits_{x\to 0_+}\frac{1}{x}(1-\cos\sqrt{x}) \;/\cdot \frac{(1+\cos\sqrt{x})}{(1+\cos\sqrt{x})} \Rightarrow$

$\Rightarrow -\lim\limits_{x\to 0_+}\frac{1}{x}\underbrace{\frac{1-\cos^2\sqrt{x}}{(1+\cos\sqrt{x})}}_{\to 2} = -\lim\limits_{x\to 0_+}\frac{1}{2x}\cdot\sin^2\sqrt{x} = -\lim\limits_{x\to 0_+}\frac{1}{2x}(\sin\sqrt{x})^2 =$

$= -\lim\limits_{x\to 0_+}\frac{1}{2x}(\sqrt{x})^2 = -\lim\limits_{x\to 0}\frac{x}{2x} = -\frac{1}{2}$

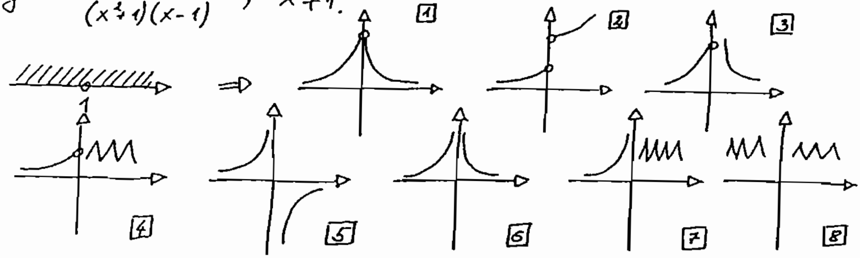
$\ln L = -\frac{1}{2} \Rightarrow L = e^{-\frac{1}{2}} \Rightarrow L = \frac{1}{\sqrt{e}}$

* koristili smo smjenu (ekvivalenciju)

$\begin{cases}\sin\sqrt{x} \sim \sqrt{x}, \\ \sqrt{x}\to 0\end{cases}$ prema $\begin{cases}\sin x \sim x, \\ x\to 0\end{cases}$

(Z) Da li f-ja f(x) ima graničnu vrijednost u $x=1$?

$f(x) = \frac{x(x^2-1)}{(x^2+1)(x-1)}, \quad x \neq 1.$



$L = \lim\limits_{x\to 1}\frac{x(x-1)(x+1)}{(x^2+1)(x-1)} = \lim\limits_{x\to 1}\frac{x(x+1)}{x^2+1} = \lim\limits_{x\to 1}\frac{x^2+x}{x^2+1} = \frac{2}{2} = 1$

Ima konačnu graničnu vrijednost : sliko 1.

(Z) Iz familije $f(x,\alpha) = \begin{cases}\alpha x+1, & x<1 \\ \sin x\pi + 2, & x>1\end{cases}$ izdvojiti samo onu f-ju koja u $x=1$ ima konačnu graničnu vrijednost!

$L_1 = \lim\limits_{x\to 1_-} f(x) = \lim\limits_{x\to 1\uparrow} \alpha x+1 = \alpha+1$

$L_2 = \lim\limits_{x\to 1_+} f(x) = \lim\limits_{x\to 1\downarrow} (\sin x\pi + 2) = 2$

Da bi f-ja imala konačnu gr. vr. mora biti: $L_1 = L_2$ pa je:

$\alpha+1 = 2 \Rightarrow \alpha = 1 \Rightarrow$ Tražena f-ja je:

$f(x) = \begin{cases} x+1, & x<1 \\ \ldots \end{cases}$

* Definicija: Data f-ja $f(x)$ u točki $x=a$ ima (nema) konačnu ili ∞ gr. vrijednost a, (L_1 ne postoji) tj, imamo slučajeve:

$$\lim_{x\to a} f(x) = const \Rightarrow \text{ima gr. vrijednost.}$$
$$= \infty \Rightarrow \text{ima } \infty \text{ graničnu vrijednost}$$
$$= \text{nepostoji} \Rightarrow \text{nema granične vrijednosti}$$

(Z) Da li $f(x) = \dfrac{1}{1+2^{\frac{1}{x}}}$, $x \neq 0$ ima gr. vrijednost u $x=0$?

$$L_1 = \lim_{x\to 0_-} f(x) = \lim_{x\to 0_-} \frac{1}{1+2^{\frac{1}{0_-}}} = \left| 2^{-\infty} = \frac{1}{2^{\infty}} = \frac{1}{\infty} = 0 \right| = \underline{1}.$$

$$L_2 = \lim_{x\to 0_+} f(x) = \lim_{x\to 0_+} \frac{1}{1+2^{1/0_+}} = \underline{0}.$$

kako je $L_1 \neq L_2$ f-ja divergira u širem smislu (u.š.s.), a što se dešava kada se nađu L_1, L_2, ali budu različiti.

(Z) $f(x) = \sin\frac{1}{x}$, $x \neq 0$. Da li ova f-ja u točki $x=0$ ima graničnu vrijednost?

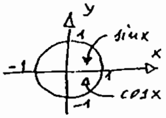
$D(f)$: $\forall\, x \setminus \{0\} \in R$ tj. (brojna os bez točke 0)

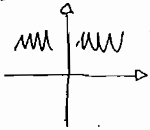
$$L_2 = \lim_{x\to 0_+} \sin\frac{1}{x} = \left| \frac{1}{x} = 2n\pi + \alpha,\ \alpha \in R,\ n \in N \Rightarrow x = \frac{1}{2n\pi+\alpha},\ \begin{matrix} x\to 0_+ \\ n\to\infty \end{matrix} \right| =$$

$$= \lim_{n\to\infty} \sin(2n\pi + \alpha) = \left| \sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta \right| =$$

$$= \lim_{n\to\infty} (\sin 2\pi n \cdot \cos\alpha + \cos 2\pi n \cdot \sin\alpha) = \lim_{n\to\infty} \sin\alpha = \underline{\sin\alpha}.$$

jer je: $\begin{cases} \sin 0 = 0 \\ \cos 0 = 1 \\ \sin 2\pi = 0 \\ \cos 2\pi = 1 \quad \text{itd...} \end{cases}$





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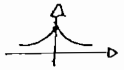
$$\sin\alpha = \begin{cases} 0, & \alpha = 0 \\ 1, & \alpha = \pi/2 \\ \vdots & \vdots \end{cases} \Rightarrow \left. \begin{matrix} L_1 \text{ (D) u.š.s.} \\ L_2 \text{ (D) u.š.s.} \end{matrix} \right\} \Rightarrow \text{nema granične vrijednosti.}$$

* Data f-ja u točki tipo (brojna os, $x=a$) je neprekidna u $x=a$ ako je:

$$\lim_{x\to a} f(x) = f(a)$$



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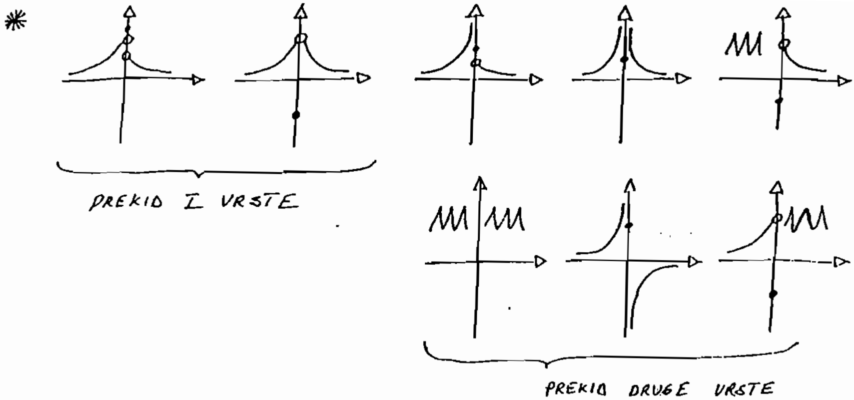


(Z) Iz familije $f(x,\alpha) = \begin{cases} \alpha x^2 + x + 3, & x \le 1 \\ \sin\frac{\pi x}{2} + 2\alpha, & x > 1 \end{cases}$ izdvojiti neprekidnu f-ju!

* za $\forall x \neq 1 \Rightarrow$ neprekidna f-ja (familija f-ja) $\forall \alpha$.

* za $x = 1$ mora biti $L_1 = L_2 = L = f(1)$

$$\left.\begin{array}{l} L_1 = \lim\limits_{x \to 1_-} (\alpha x^2 + x + 3) = \alpha + 4 \\ L_2 = \lim\limits_{x \to 1_+} \left(\sin\frac{\pi x}{2} + 2\alpha\right) = 1 + 2\alpha \end{array}\right\}$$

$L_1 = L_2 \Rightarrow$ ovo mora biti da bi dobili neprekidnu f-ju.

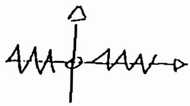
$\alpha + 4 = 1 + 2\alpha \Rightarrow \underline{\alpha = 3}!$

$$f(x) = \begin{cases} 3x^2 + x + 3, & x \le 1 \longrightarrow f(1) = 7 \\ \sin\frac{\pi x}{2} + 6, & x > 1 \longrightarrow f(1) = 7 \end{cases}$$

(Z) Za koje $n \in \mathbb{N}$ je f-ja $f(x)$ data sa:

$$\left.\begin{array}{l} f(x) = x^{n-1} \sin\frac{1}{x}, \ x \neq 0 \\ f(0) = 0 \end{array}\right\} \text{ neprekidna u } x = 0 ?$$

Za $n = 1 \Rightarrow$ $f(x) = x^0 \sin\frac{1}{x} = \sin\frac{1}{x}$; $\lim\limits_{x \to 0} \sin\frac{1}{x} \Rightarrow$ (D) UŠS.



$f(0) = 0$ pa za $n = 1$ f-ja ima prekid II vrste.

za $n = 2, 3, 4, \ldots \Rightarrow \lim\limits_{x \to 0} x^{n-1} \sin\frac{1}{x} = 0$ iz

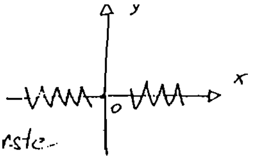
$0 < \left|x^{n-1} \sin\frac{1}{x}\right| < |x|^{n-1} \Big/ \cdot \lim\limits_{x \to 0}, \ n = 2, 3, \ldots \Rightarrow$

$0 \le \lim\limits_{x \to 0} \left|x^{n-1} \sin\frac{1}{x}\right| \le \lim\limits_{x \to 0} |x|^{n-1} = 0 \Rightarrow 0 \le \lim\limits_{x \to 0} \left|x^{n-1} \sin\frac{1}{x}\right| \le 0 \Rightarrow$

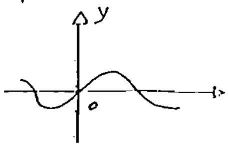
$\lim\limits_{x \to 0} \left|x^{n-1} \sin\frac{1}{x}\right| = 0$ pa je: $L = \lim\limits_{x \to 0} x^{n-1} \sin\frac{1}{x} = 0 = f(0) = 0$ pa je

za $n = 2, 3, 4, \ldots$ data f-ja neprekidna u $x = 0$.

Dakle, imamo: za $n=1$ prekid druge vrste u $x=0$.



za $n=2,3,4,\dots$ neprekidno u $x=0$.



(2) $f(x) = \ln\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}, \quad x \neq 0$; $f(0) = \alpha$ } za koje je α f-ja neprekidna u $x=0$?

Mora biti $\alpha = L$.

$$L = \lim_{x\to 0} \ln\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} \overset{0/0}{=} |Z.R.T| = \frac{1}{x^2}\cdot \ln\left(\frac{\sin x}{x}\right) =$$

$$= \left|\begin{matrix}\ln u \sim u-1\\ u\to 1\end{matrix}\right| = \left|\lim_{u\to 1}\frac{\ln u}{u-1} = 1\right| = \lim_{x\to 0} \underbrace{\frac{\ln\frac{\sin x}{x}}{\left(\frac{\sin x}{x}-1\right)}}_{\to 1} \frac{\left(\frac{\sin x}{x}-1\right)}{x^2} =$$

$$= \lim_{x\to 0}\frac{\frac{\sin x}{x}-1}{x^2} = -\lim_{x\to 0}\frac{x-\sin x}{x^3} =$$

$$= -\lim_{x\to 0}\frac{x-\sin x}{x^3} = -\frac{1}{6}$$

$\alpha = -\frac{1}{6}$. Za $\alpha = -\frac{1}{6}$ f-ja je neprekidna u tački $x=0$.

*

$$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \overset{0/0}{=} L' \quad \text{ili uz} \quad x-x_0=\Delta x \Rightarrow$$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \overset{0/0}{=} L'$$

Ako je $L' = \pm\infty$ imaćemo ∞ izvod } izuzimamo se
Ako je L' = (D) š.S. – nemamo izvoda } $f(x) = const.$

(Z) $\left.\begin{array}{l} f(x)=x^2 \\ x_0=1 \end{array}\right\} \forall x \wedge f(1)=1.$ Naći izvod u točki $x=1$.

$$\frac{f(x)-f(1)}{x-1} = \frac{x^2-1}{x-1}, \quad x \neq 1$$

$$\lim_{\substack{x\to 1\\ x\neq 1}} \frac{f(x)-f(1)}{x-1} = \lim_{\substack{x\to 1\\ x\neq 1}} \frac{(x-1)(x+1)}{(x-1)} = 2.$$

Izvod f-je $f(x)=x^2$ u $x=1$ jednak je 2.

(Z) $f(x)=\ln x \Rightarrow f'(x)=\frac{1}{x}\Big|_{x=e} = \frac{1}{e}$

$$\frac{f(x)-f(e)}{x-e} = \frac{\ln x-1}{x-e}, \quad x\neq e$$

$$\lim_{\substack{x\to e\\ x\neq e}} \frac{f(x)-f(e)}{x-e} = \lim_{x\to e} \frac{\ln x-1}{x-e} \overset{\frac{0}{0}}{=} \left| \begin{array}{l} x-e=t,\ x=t+e \\ x\to e,\ t\to 0 \end{array}\right| =$$

$$= \lim_{t\to 0} \frac{\ln(t+e)-\ln e}{t} = \lim_{t\to 0} \frac{\ln\frac{t+e}{e}}{t} = \lim_{t\to 0} \underbrace{\left(\frac{\ln(1+\frac{t}{e})}{\frac{t}{e}}\right)}_{\to 1} \cdot \frac{1}{e} = \frac{1}{e}.$$

* $\ln(1+x)$, kad $x\to 0$ mijenjamo sa e^x-1, kad $x\to 0$.

$$\lim_{x\to 0} (1+x)^{\frac{1}{x}} = e \ \Big/ \cdot \ln \Rightarrow \lim_{x\to 0} \frac{1}{x}\cdot \ln(1+x) = \lim_{x\to 0} \frac{e^x-1}{x} = 1$$

(Z) Odrediti inverznu f-ju f-je date sa:

$y=\ln\left(\arcsin\frac{x}{x+1}\right)$!

Najprije tražimo definiciono područje:

$$1^\circ \quad \left.\begin{array}{l} \arcsin\frac{x}{x+1} > 0 \\ \left|\frac{x}{x+1}\right| \leq 1 \end{array}\right\} \Rightarrow 0 < \underbrace{\frac{x}{x+1} \leq 1}_{\frac{x}{x+1}-1\leq 0} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} 0 < \frac{x}{x+1} \\ -\frac{1}{x+1} \leq 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} \frac{x}{x+1} > 0 \\ x+1 > 0 \end{array}\right\} \Rightarrow \underline{x>0 : \mathcal{D}_x}.$$

2° Zatim treba pokazati da je $f(x)$ monotona f-ja:

$$x_2 > x_1 \Rightarrow \arcsin\frac{x_2}{x_2+1} > \arcsin\frac{x_1}{x_1+1} \Rightarrow \frac{x_2}{x_2+1} > \frac{x_1}{x_1+1} \Rightarrow$$

$$x_2\cdot(x_1+1) > x_1(x_2+1) \Rightarrow x_1x_2 + x_2 > x_1x_2 + x_1 \Rightarrow \underline{x_2 > x_1}.$$

F-ja je monotono rastuća.

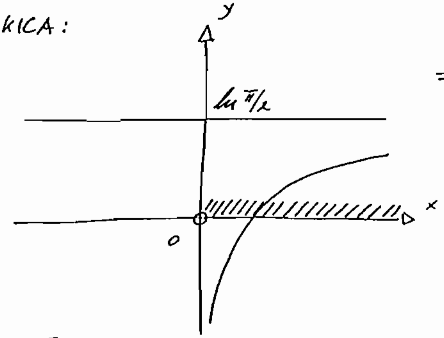
$*$ $$\lim_{x\to+\infty} \ln\arcsin\frac{x}{x+1} = \ln \arcsin \lim_{x\to\infty}\frac{x}{x+1} = \ln\arcsin\lim_{x\to\infty}\frac{1}{1+\frac{1}{x}} =$$

$$= \ln\underbrace{\arcsin 1}_{\pi/2} = \ln\frac{\pi}{2} > 1.$$

$$\lim_{x\to 0_+} \ln\arcsin\frac{x}{x+1} = \ln\arcsin\lim_{x\to 0_+}\frac{0}{0+1} = \ln\underbrace{\arcsin 0}_{0} =$$

$$= \ln 0 = -\infty.$$

SKICA:



$$\Rightarrow E_y:\ y \in (-\infty, \ln \pi/2)$$

$*$ Traženje inverzije:

$$\arcsin\frac{y}{y+1} = e^x,\quad \frac{y}{y+1} = \sin e^x \Rightarrow y = y\sin e^x + \sin e^x$$

RJ: $$\begin{cases} y = \dfrac{\sin e^x}{1-\sin e^x}, & y > 0 \\ -\infty < x < \ln\frac{\pi}{2}. \end{cases}$$ (B)

(2) Dokazati da je $\lim\limits_{x\to+\infty}\frac{4x-3}{2x+1}=2$!

$$|f(x)-A| = \Delta = \left|\frac{4x-3}{2x+1}-2\right| = \frac{|4x-3-4x-2|}{|2x+1|} = \frac{5}{2x+1} = \Delta,\ x>0$$

* Potrebno je moći Δ-tu učiniti po volji malom kada manipulišemo sa $x>0$.

$\Delta = \frac{5}{2x+1} < \frac{5}{2x} < \varepsilon \Rightarrow$ Neka je ε neka konstanta. kakav treba biti x da Δ bude manji od ε?

$$\frac{1}{x} = \frac{2}{5}\varepsilon \Rightarrow \underline{x = \frac{5}{2\varepsilon}}.$$

Tehnika rješavanja Limesa (II dio)

* Granični procesi ili limesi mogu da budu sa x ili sa n. Također se dijele na trivijalne i složene! Šema objašnjava postupak rješavanja:

$\lim\limits_{x\to\infty} f(x)$ | x / n

→ trivijalni ⇒ direktno očitanje ⇒ STOP.

→ složeni ⇒ P.N.O. (prividno neodređeni): $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, \infty^0, 1^\infty, 0^0 \Rightarrow$

⇒ Z.R.T. (zahtjevaju regularnu transformaciju) i svode se na trivijalne.

AKSIOM 1. (INVARIJANTA)

$$\lim_{x\to a} f(x) = \lim_{z\to a} f(z)$$

AKSIOM 2. (OČITANJA)

$$\lim_{\substack{x\to a\\ x\neq a}} f(x) = A = \lim_{x\to a} A \quad , (\text{NIKAO „}A\text{"})$$

AKSIOM 3. (PRIORITETA)

$$\lim_{\substack{x\to a\\ \lambda\to\lambda_0}} f(x,\lambda) = \lim_{x\to a} f(x,\lambda_0)$$

AKSIOM 4. (HAJNEOVA SMJENA)

$$\lim_{x\to a} f(x) = \left| \begin{array}{l} x=\varphi(t), \quad t\to t_0 \text{ kod } x\to a \\ x=\varphi(t) - \text{mora biti } \text{Ⓑ}!!! \end{array} \right| = \lim_{t\to t_0} f(\varphi(t))$$

AKSIOM 5. (OPŠTI HAJNEOV AKSIOM)

$$\lim_{x\to a} f(x) = \left| \begin{array}{l} x-a = \alpha_n^{(i)} \xrightarrow[(n)]{} 0 \\ \forall i \text{ (izbor nula niza)} \\ \alpha_n^{(i)} > \alpha_{n+1}^{(i)} \end{array} \right| = \lim_{\alpha_n\to 0} f(a+\alpha_n^{(i)})$$

AKSIOM 6. (BITAN KOD F-JA VIŠE PROMJENJIVIH)

$$\left.\begin{array}{l} \lim_{x\to a} f(x) = A \\ a=\text{const.} \quad A=\text{const.} \end{array}\right\} \sim \begin{array}{l} 0<|x-a|<\delta \Rightarrow |f(x)-A|<\varepsilon \\ \delta\to 0_+ \\ \varepsilon\to 0_+ \end{array}$$

AKSIOM 7. (AKSIOM EKVIVALENCIJE I POREĐENJA)

$$* \quad \lim_{x\to a} \frac{\varphi(x)}{h(x)} \Rightarrow \varphi(x) \sim h(x), \quad x\to a$$

$$* \quad \varphi(x)>0, \quad h(x)>0 \quad \wedge \quad \lim_{x\to a} \frac{\varphi(\cdot)}{h(x)} = 1 \Rightarrow \varphi(x) < 2h(x)$$

$$x\to a \qquad \text{tj.} \quad x\in O^{(a)}$$

⁕ Upotreba tehnike Limesa :

$\lim_{x \to a} f(x)$

- → Granična vrijednost f-je u tački $x=a$.
- → Neprekidnost f-je.
- → Izvod f-je u tački, intervalu itd,...
- → Određeni integral

(Z) Naći Limes : $\lim_{x \to 0_+} \frac{e^{a^x}}{x^k} = L$, $k > 0$, $a \neq 0$

* $0 < a \neq 1$ - ovo je uslov kod eksponencijalne f-je

(1) $0 < a < 1 \Rightarrow L = \lim_{x \to 0_+} \frac{e^{a^x}}{x^k} = \lim_{x \to 0_+} \frac{e}{x^k} = +\infty$

(2) $a = 1 \Rightarrow L = \lim_{x \to 0_+} \frac{e^1}{x^k} = +\infty$

(3) $a > 1 \Rightarrow L = \lim_{x \to 0_+} \frac{e^{a^x}}{x^k} = +\infty$

(Z) $\lim_{x \to +\infty} \frac{500x + 10^4}{0{,}001x^2 + 1} \overset{\frac{\infty}{\infty}}{=} \lim_{x \to +\infty} \frac{\frac{500}{x} + \frac{10^4}{x^2}}{0{,}001 + \underbrace{\frac{1}{x^2}}_{\to 0}} = \frac{0}{0{,}001 + 0} = 0$

$\to 0$ jer je moguće $\frac{1}{x^2}$ učiniti još manjim za $x \gg$.

(Z) $L = \lim_{x \to -1} \frac{1 + \sqrt[5]{x}}{1 + \sqrt[3]{x}}$

$\lim_{x \to -1} \frac{1 + \sqrt[5]{x}}{1 + \sqrt[3]{x}} \overset{\frac{0}{0}}{=} |Z.R.T| = \left| \begin{matrix} x = t^{15} \\ t \to -1 \text{ (B)} \end{matrix} \right| = \lim_{t \to -1} \frac{1 + t^3}{1 + t^5} \overset{\frac{0}{0}}{=} |Z.R.T| =$

$= \left| \begin{matrix} t = -x \\ x \to 1 \text{ (B)} \end{matrix} \right| = \lim_{x \to 1} \frac{x^3 - 1}{x^5 - 1} \overset{\frac{0}{0}}{=} |\text{BINOMNI OBRAZAC}| =$

$= \left| \frac{(x-1)(x^2 + x + 1)}{(x-1)(x^4 + x^3 + x^2 + x + 1)} \right| \odot \lim_{x \to 1} \Rightarrow$

$\Rightarrow \lim_{x \to 1} \frac{x^2 + x + 1}{x^4 + x^3 + x^2 + x + 1} = \frac{3}{5}$

Ovdje ste morali znati formulu :

$a^{2n} - b^{2n} = (a^n - b^n)(a^n + b^n)$; $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

$a^{2n+1} + b^{2n+1} = (a+b)(a^{2n} \mp a^{2n-1}b + a^{2n-2}b^2 \mp \dots + a^2b^{2n-2} \mp ab^{2n-1} \pm b^{2n})$

II način:

$$L = \lim_{x\to -1} \frac{1+\sqrt[5]{x}}{1+\sqrt[3]{x}} \overset{\frac{0}{0}}{=} \lim_{x\to -1} \frac{1+x^{\frac{1}{5}}}{1+x^{\frac{1}{3}}} = \lim_{x\to -1} \frac{-x^{\frac{1}{5}}-1}{-x^{\frac{1}{3}}-1} = \left| \text{dignemo na e} \right| =$$

$$= \lim_{x\to -1} \frac{e^{\frac{1}{5}\ln(-x)}-1}{e^{\frac{1}{3}\ln(-x)}-1} \overset{\frac{0}{0}}{=} \left| \begin{matrix} e^{\alpha}-1 \sim \alpha \\ \alpha \to 0 \end{matrix} \right| = \lim_{x\to -1} \frac{\frac{1}{5}\ln(-x)}{\frac{1}{3}\ln(-x)} = \underline{\frac{3}{5}}.$$

* $A^{\alpha} = e^{\ln A^{\alpha}} = e^{\alpha \ln A}$

* $\lim_{x\to a} \varphi(x) \circ \psi(x) = L \wedge \varphi(x) \sim h(x) \Rightarrow$

$\Rightarrow L = \lim_{x\to a} h(x) \circ \psi(x)$

* Upotrebljene su ekvivalencije!

Tablica ekvivalencija

EKVIVALENCIJE I REDA:

1. $(1+x)^{\frac{1}{x}} \sim e$, $x\to 0$
2. $\ln(1+x) \sim x$, $x \to 0$
3. $\ln x \sim x-1$, $x\to 1$
4. $\operatorname{sh} x = \frac{e^x - e^{-x}}{2} \sim x$, $x \to 0$
5. $e^x - 1 \sim x$, $x \to 0$
6. $e^{f(x)} - 1 \sim f(x)$, $f(x) \xrightarrow[\text{(B)}]{x\to a} 0$
7. $e^{\sin x} - 1 \sim \sin x$, $x \to 0$
8. $\sin x \sim x$, $x \to 0$
9. $\arcsin x \sim x$, $x \to 0$
10. $\operatorname{tg} x \sim x$, $x \to 0$
11. $\operatorname{arctg} x \sim x$, $x \to 0$

EKVIVALENCIJE II REDA:

1. $\sin f(x) \sim f(x)$, $f(x) \xrightarrow[\text{(B)}]{x\to a} 0$
2. $\sin x \sim x - \frac{x^3}{6}$, $x \to 0$
3. $e^x \sim 1 + x + \frac{x^2}{2}$, $x \to 0$
4. $\cos x \sim 1 - \frac{x^2}{2}$, $x \to 0$

(Z) $L=\lim\limits_{x\to 0}\dfrac{\sqrt[n]{1+x\sin 3x}\cdot\sqrt[n]{1+\operatorname{tg}2x}-1}{e^{\sin x}-1}$

$L\overset{\frac{0}{0}}{=}\lim\limits_{x\to 0}\dfrac{(1+x\sin 3x)^{\frac{1}{n}}\cdot(1+\operatorname{tg}2x)^{\frac{1}{n}}-1}{e^{\sin x}-1}=\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{n}\ln(1+x\sin 3x)+\frac{1}{n}\ln(1+\operatorname{tg}2x)}-1}{e^{\sin x}-1}=$

$=\left|\begin{matrix}e^u-1\sim u\\ u\to 0\end{matrix}\right|=\lim\limits_{x\to 0}\dfrac{e^{\frac{1}{n}\ln(1+x\sin 3x)+\frac{1}{n}\ln(1+\operatorname{tg}2x)}-1}{\sin x}=\left|\begin{matrix}\sin u\sim u\\ u\to 0\end{matrix}\right|=$

$=\lim\limits_{x\to 0}\dfrac{\frac{1}{n}\left(\ln(1+x\sin 3x)+\ln(1+\operatorname{tg}2x)\right)}{x}=\left|\begin{matrix}\ln(1+u)\sim u,\\ u\to 0\end{matrix}\right|=$

$=\lim\limits_{x\to 0}\dfrac{\frac{1}{n}(x\sin 3x+\operatorname{tg}2x)}{x}=\dfrac{1}{n}\lim\limits_{x\to 0}\dfrac{x\sin 3x+\operatorname{tg}2x}{x}\overset{\frac{0}{0}}{=}$

$=\dfrac{1}{n}\lim\limits_{x\to 0}\sin 3x+\dfrac{1}{n}\lim\limits_{x\to 0}\dfrac{\operatorname{tg}2x}{x}=0+\dfrac{1}{n}\lim\limits_{x\to 0}\dfrac{\operatorname{tg}2x}{x}=\left|\begin{matrix}\operatorname{tg}u\sim u\\ u\to 0\end{matrix}\right|=$

$=\dfrac{1}{n}\lim\limits_{x\to 0}\dfrac{2x}{x}=\dfrac{2}{n}\Rightarrow\quad \underline{L=\dfrac{2}{n}}.$

(Z) Odrediti red beskonačno male veličine

$f(x)=1-\cos x,\ x\to 0.$

Traženi red se dobije iz uslova:

$$L\overset{\frac{0}{0}}{=}\lim_{x\to x_{0+}}\frac{f(x)}{(x-x_0)^{\alpha}}=const\neq 0.$$

$L=\lim\limits_{x\to 0_+}\dfrac{1-\cos x}{x^{\alpha}}=const\neq 0$

$L=\lim\limits_{x\to 0_+}\dfrac{2\sin^2\frac{x}{2}}{x^{\alpha}}=\left|\text{iz: }\sin\frac{\alpha}{2}=\pm\sqrt{\frac{1-\cos\alpha}{2}}\Rightarrow 1-\cos\alpha=2\cdot\sin^2\frac{\alpha}{2}\right|=$

$=\left|\begin{matrix}\text{uz: }\sin u\sim u\\ u\to 0\end{matrix}\right|=\lim\limits_{x\to 0_+}\dfrac{2\cdot\frac{x^2}{4}}{x^{\alpha}}=\lim\limits_{x\to 0_+}\dfrac{x^2}{2\cdot x^{\alpha}}=\dfrac{1}{2}\lim\limits_{x\to 0_+}\dfrac{x^2}{x^{\alpha}}$

$\Rightarrow\quad L=\dfrac{1}{2}\lim\limits_{x\to 0_+}\dfrac{x^2}{x^{\alpha}}\Rightarrow$ imamo slučajeve:

za $\alpha=1\Rightarrow\dfrac{1}{2}\lim\limits_{x\to 0_+}\dfrac{x^2}{x}=\dfrac{1}{2}\lim\limits_{x\to 0_+}x=0$ [ne]

za $\left.\begin{array}{l}\alpha<1\\ \alpha>0\end{array}\right\} \Rightarrow \frac{1}{2}\lim_{x\to 0_+}\frac{x^2}{\sqrt[\alpha]{x}} = \left|\begin{array}{l}\text{dijelim sa}\\ \text{najvećim}\\ \text{stepenom}\end{array}\right| =$

$$= \frac{1}{2}\lim_{x\to 0_+}\frac{1}{x^{\alpha-2}} = 0 \qquad \boxed{ne}$$

za $\alpha = 0 \Rightarrow \frac{1}{2}\lim_{x\to 0_+}\frac{x^2}{1} = 0 \qquad \boxed{ne}$

za $\left.\begin{array}{l}\alpha>1\\ \alpha<2\end{array}\right\} \Rightarrow \frac{1}{2}\lim_{x\to 0_+}\frac{x^2}{x^\alpha} \Big/ : x^2 = \frac{1}{2}\lim_{x\to 0_+}\frac{1}{x^{\alpha-2}} = 0 \quad \boxed{ne}$

za $\alpha = 2$. ŠTO JE I BILO OČIGLEDNO:

$$L = \frac{1}{2}\lim_{x\to 0_+} 1 = \frac{1}{2} \Rightarrow$$

$\Rightarrow \quad 1-\cos x = \frac{x^2}{2}$ tj. $1-\cos x \sim \frac{x^2}{2},\ x\to 0_+$.

zbog limesa „nikad neće biti".

② Odrediti red beskonačno male veličine:

$\varphi(x) = \operatorname{sh}x - x,\ x\to 0.$

kako smo već rekli, ovo se radi prema formuli:

$$L \overset{0}{\underset{0}{=}} \lim_{x\to x_{0+}}\frac{f(x)}{(x-x_0)^\alpha} = const \neq 0.$$

$*\ \operatorname{sh}x = \frac{e^x - e^{-x}}{2}.$

$$\operatorname{sh}2x = 2\operatorname{sh}x\cdot\operatorname{ch}x \Rightarrow \frac{e^{2x}-e^{-2x}}{2} = \frac{e^x-e^{-x}}{2}\cdot\frac{e^x+e^{-x}}{2}\cdot 2 =$$

$$= \frac{(e^x-e^{-x})(e^x+e^{-x})}{2}$$

$$L = \lim_{x\to 0_+}\frac{\operatorname{sh}x - x}{x^\alpha} = const \neq 0.$$

$$\lim_{x\to 0_+}\frac{\operatorname{sh}x - x}{x^\alpha} = \left|\begin{array}{l}x = 2t\\ t\to 0,\ \text{Ⓑ}\end{array}\right| = \lim_{t\to 0}\frac{\operatorname{sh}2t - 2t}{(2t)^\alpha} =$$

$$= \lim_{t\to 0} \frac{2\,sh t \cdot ch t - 2t}{(2t)^\alpha} = 2\cdot\lim_{t\to 0}\frac{sh t\, ch t - t}{(2t)^\alpha} = \left| \begin{matrix}\text{dodamo i}\\ \text{oduzmemo}\\ sh t\end{matrix}\right| =$$

$$= 2\cdot\lim_{t\to 0}\frac{sh t - sh t + sh t\, ch t - t}{(2t)^\alpha} = 2\cdot\lim_{t\to 0}\frac{sh t + sh t(ch t - 1) - t}{(2t)^\alpha} =$$

$$= 2\cdot\lim_{t\to 0}\frac{sh t(ch t - 1)}{t\cdot 2^\alpha \cdot t^{\alpha-1}} + 2\cdot\lim_{t\to 0}\frac{sh t - t}{(2t)^\alpha} =$$

$$= 2\cdot\lim_{t\to 0}\left[\frac{sh t}{t}\cdot\frac{(ch t - 1)}{2^\alpha t^{\alpha-1}} + \frac{sh t - t}{(2t)^\alpha}\right] = \left| \begin{matrix} t = x \\ x\to 0\end{matrix}\right| =$$

$$= \frac{2}{2^\alpha}\lim_{t\to 0}\left[\frac{ch t - 1}{t^{\alpha-1}} + \frac{sh t - t}{t^\alpha}\right] = \frac{1}{2^{\alpha-1}}\lim_{x\to 0}\left[\frac{ch x - 1}{x^{\alpha-1}} + \frac{sh x - x}{x^\alpha}\right] =$$

$$= \left| ch x - 1 = 2 sh^2 \frac{x}{2}\right| = \frac{1}{2^{\alpha-1}}\lim_{x\to 0}\Bigg[\underbrace{\frac{2 sh^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}}_{\to 2}\cdot\frac{\left(\frac{x}{2}\right)^2}{x^{\alpha-1}} + \underbrace{\frac{sh x - x}{x^\alpha}}_{L_0}\Bigg] =$$

$$= \frac{1}{2^{\alpha-1}}\lim_{x\to 0}\left[\frac{2\cdot x^2}{4\cdot x^{\alpha-1}} + L_0\right] = \frac{1}{2^{\alpha-1}}\lim_{x\to 0}\left[\underbrace{\frac{x^2}{2x^{\alpha-1}}} + L_0\right] =$$

Odavde slijedi da $\underbrace{\alpha - 1 = 2}_{\text{mora biti!}} \Rightarrow$

$\Rightarrow \underline{\alpha = 3}$.

$$= \frac{1}{4}\cdot\left[\frac{1}{2} + L_0\right] = \frac{1}{8} + \frac{L_0}{4} \Rightarrow$$

$$\Rightarrow L_0 = \frac{1}{8} + \frac{L_0}{4} \Rightarrow 4L_0 - L_0 = \frac{1}{2} \Rightarrow 3L_0 = \frac{1}{2} \Rightarrow \underline{L_0 = \frac{1}{6}}.$$

Dakle, $\left.\begin{matrix}\alpha = 3\\ L = \frac{1}{6}\end{matrix}\right\}\ \varphi(x) = sh x - x \sim \frac{x^3}{6},\ \underline{x\to 0}.$

$*\quad sh x \sim 0,\ x\to 0$

$\quad sh x \sim x,\ x\to 0$

$\quad sh x \sim x + \frac{x^3}{6},\ x\to 0$

$\quad sh x \sim x + \frac{x^3}{3!} + \frac{x^5}{5!},\ x\to 0$

$\quad sh x \sim x + \frac{x^3}{3!} + \ldots + \frac{x^{2n+1}}{(2n+1)!},\ x\to 0$

$\lim f(x) = A \iff f(x) = A + \varepsilon,\ \varepsilon \to 0,\ x \to 0$

$\operatorname{sh} x = x + \frac{x^3}{3!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + o(x^{2n-3}),\ x \to 0$

$$\lim_{x \to 0} \frac{\operatorname{sh} x - x - \frac{x^3}{6}}{x^5} = L$$

$$L = \lim_{x \to 0} \frac{x + \frac{x^3}{6} + \frac{x^5}{5!} + o(x^7)}{x^5} = \lim_{x \to 0} \left(\frac{1}{120} + o(x^2)\right) = \frac{1}{120}$$

Imamo ekvivalencije 0-tog reda i prvog reda koje se najviše koriste.

Sve što je rođeno, rođeno je na osnovu slova.

$\lim_{x \to a} f(x) = L \wedge L = \alpha L + \beta,\ \alpha \neq 1$ tada je: $L = \frac{\beta}{1-\alpha}$

(Z) $$L = \lim_{x \to 0} \frac{\sqrt{1 - \operatorname{tg} x} - \sqrt{1 + \operatorname{tg} x}}{\sin 2x}$$

$$\lim_{x \to 0} \frac{\sqrt{1 - \operatorname{tg} x} - \sqrt{1 + \operatorname{tg} x}}{\sin 2x} \overset{\frac{0}{0}}{=} \left| \begin{array}{l} \sqrt{\alpha} - \sqrt{\beta} = \frac{\alpha - \beta}{\sqrt{\alpha} + \sqrt{\beta}},\ \alpha > 0,\ \beta > 0 \quad \sqrt[3]{\alpha} - \sqrt[3]{\beta} = \frac{\alpha - \beta}{\sqrt[3]{\alpha^2} + \sqrt[3]{\alpha \cdot \beta} + \sqrt[3]{\beta^2}} \\ \sin 2x = 2 \sin x \cos x \end{array} \right| =$$

$$= \left| \sqrt[3]{\alpha} - \sqrt[3]{\beta} = \frac{\alpha - \beta}{\sqrt[3]{\alpha^2} + \sqrt[3]{\alpha \cdot \beta} + \sqrt[3]{\beta^2}} \right| = \lim_{x \to 0} \frac{\frac{1 - \operatorname{tg} x - 1 - \operatorname{tg} x}{\sqrt{1 + \operatorname{tg} x} + \sqrt{1 - \operatorname{tg} x}}}{2 \sin x \cos x} =$$

$$= \lim_{x \to 0} \frac{-2 \operatorname{tg} x}{2 \sin x \cos x \left[\sqrt{1 + \operatorname{tg} x} + \sqrt{1 - \operatorname{tg} x}\right]} = \lim_{x \to 0} \frac{-\frac{\sin x}{\cos x}}{\sin x \cdot \cos x \underbrace{\left[\sqrt{1 + \operatorname{tg} x} + \sqrt{1 - \operatorname{tg} x}\right]}_{\to 2}} =$$

$$= \lim_{x \to 0} \frac{-\sin x}{2 \sin x \cos x \cdot \cos x} = -\lim_{x \to 0} \frac{1}{\underbrace{2 \cdot \cos^2 x}_{\to 1}} = -\frac{1}{2}$$

(Z) $L = \lim_{x\to 0} \frac{\sqrt[31]{1-\operatorname{tg}x} - \sqrt[163]{1+\operatorname{tg}x}}{\sin 3x} \overset{0}{\underset{0}{=}} \lim_{x\to 0} \frac{(1-\operatorname{tg}x)^{\frac{1}{31}} - (1+\operatorname{tg}x)^{\frac{1}{163}}}{\sin 3x} \overset{0}{\underset{0}{=}}$

$= \left| \begin{matrix} \sin u \sim u \\ u \to 0 \end{matrix} \right| = \lim_{x\to 0} \frac{(1-\operatorname{tg}x)^{\frac{1}{31}} - (1+\operatorname{tg}x)^{\frac{1}{163}}}{3x} = \left| \begin{matrix} \text{navodimo na} \\ \text{oblik } e^u - 1 \end{matrix} \right| =$

$= \lim_{x\to 0} \left[\frac{e^{\frac{1}{31}\ln(1-\operatorname{tg}x)} - 1}{3x} - \frac{e^{\frac{1}{163}\ln(1+\operatorname{tg}x)} - 1}{3x} \right] = \left| \begin{matrix} e^u - 1 \sim u, \\ u \to 0 \end{matrix} \right| =$

$= \lim_{x\to 0} \left[\frac{1}{31} \cdot \frac{\ln(1-\operatorname{tg}x)}{3x} - \frac{1}{163} \frac{\ln(1+\operatorname{tg}x)}{3x} \right] = \lim_{x\to 0} \left[\frac{\ln(1-\operatorname{tg}x)}{31\cdot 3x} - \frac{\ln(1+\operatorname{tg}x)}{163\cdot 3x} \right] =$

$= \left| \begin{matrix} \ln(1+u) \sim u, \ u \to 0 \\ \ln(1-u) \sim -u, \ u \to 0 \end{matrix} \right| = \lim_{x\to 0} \left[\frac{-\operatorname{tg}x}{31\cdot 3x} - \frac{\operatorname{tg}x}{163\cdot 3x} \right] = \left| \begin{matrix} \operatorname{tg}u \sim u, \\ u \to 0 \end{matrix} \right| =$

$= \lim_{x\to 0} \left(\frac{-x}{31\cdot 3x} - \frac{x}{163\cdot 3x} \right) = -\frac{x}{31\cdot 3x} - \frac{x}{163\cdot 3x} = -\frac{1}{93} - \frac{1}{489}$.

(Z) Odrediti Limes : $L = \lim_{x\to\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} \right)$.

$\lim_{x\to\infty} \left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} \right) \overset{\infty-\infty}{=} \left| \text{racionalizacija} \right| =$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x+\sqrt{x+\sqrt{x}}} - \sqrt{x} \right)\left(\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x} \right)}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x} \right)} = \lim_{x\to\infty} \frac{x+\sqrt{x+\sqrt{x}} - x}{\left(\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x} \right)} =$

$= \lim_{x\to\infty} \frac{\sqrt{x+\sqrt{x}}}{\sqrt{x+\sqrt{x+\sqrt{x}}} + \sqrt{x}} = \left| \begin{matrix} \text{IZVUČEMO} \\ \sqrt{x} \end{matrix} \right| = \lim_{x\to\infty} \frac{\sqrt{x}\left(\sqrt{1+\frac{1}{\sqrt{x}}} \right)}{\sqrt{x}\left(\left(\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{\sqrt{x}\cdot x}}} \right) + 1 \right)} =$

$= \lim_{x\to\infty} \frac{\sqrt{x}\left(\sqrt{1+\frac{1}{\sqrt{x}}} \right)}{\sqrt{x}\underbrace{\left(\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{\sqrt{x}\cdot x}}} + 1 \right)}_{\to 2}} = \frac{1}{2}.$

* $\sqrt{x+\sqrt{x}} = \sqrt{x\left(1+\frac{\sqrt{x}}{x}\right)} = \left| \frac{\sqrt{x}}{x} \Big/ \cdot \frac{\sqrt{x}}{\sqrt{x}} \Rightarrow \frac{x}{\sqrt{x}\cdot x} = \frac{1}{\sqrt{x}} \right| = \sqrt{x\left(1+\frac{1}{\sqrt{x}}\right)} =$
$= \sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}}$

* $\sqrt{x+\sqrt{x+\sqrt{x}}} = \sqrt{x+\sqrt{x}\sqrt{1+\frac{1}{\sqrt{x}}}} = \sqrt{x\left(1+\frac{\sqrt{x}}{x}\sqrt{1+\frac{1}{\sqrt{x}}}\right)} =$
$= \sqrt{x\left(1+\frac{1}{\sqrt{x}}\sqrt{1+\frac{1}{\sqrt{x}}}\right)} = \sqrt{x}\cdot\sqrt{1+\sqrt{\frac{1}{x}\left(1+\frac{1}{\sqrt{x}}\right)}} = \sqrt{x}\sqrt{1+\left(\frac{1}{x}+\frac{1}{\sqrt{x}\cdot x}\right)}$.

Ⓩ Riješiti Limes:

$$L = \lim_{x\to+\infty}\left(\sin\sqrt{x+1} - \sin\sqrt{x}\right) \overset{\infty-\infty}{=} \left| \sin\alpha - \sin\beta = 2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2} \right| =$$

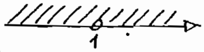
$$= \lim_{x\to+\infty} 2\cdot\sin\frac{\sqrt{x+1}-\sqrt{x}}{2}\cos\frac{\sqrt{x+1}+\sqrt{x}}{2} = 2\cdot\lim_{x\to+\infty}\sin\frac{x+1-x}{(\sqrt{x+1}+\sqrt{x})\cdot 2}\cos\frac{2x+1}{2\cdot(\sqrt{x+1}-\sqrt{x})} =$$

$$= 2\lim_{x\to\infty}\sin\underbrace{\frac{1}{2(\sqrt{x+1}+\sqrt{x})}}_{\to 0}\cdot\cos\frac{2x+1}{2\cdot(\sqrt{x+1}-\sqrt{x})} = \underline{0}.$$

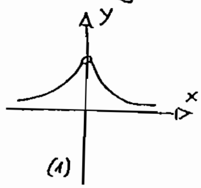
Ⓩ $L = \lim_{x\to 1}(1-x)\operatorname{tg}\frac{\pi x}{2}$ | tip $0\cdot\infty$ |

Posmatrajmo f-ju $f(x) = (1-x)\operatorname{tg}\frac{\pi x}{2}$, $x \neq 1$.

Njeno definiciono područje je:



. Ovo znači da se mogu desiti samo dva slučaja:



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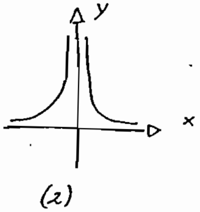
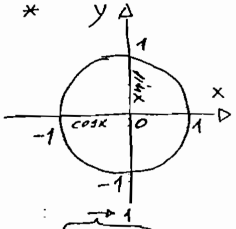




$$L \overset{0\cdot\infty}{=} \lim_{x\to 1}(1-x)\operatorname{tg}\frac{\pi x}{2} = |\text{transformacije}| = \lim_{x\to 1}(1-x)\frac{\overbrace{\sin\frac{\pi x}{2}}^{\to 1}}{\cos\frac{\pi x}{2}} =$$

$$= \lim_{x\to 1}\frac{(1-x)}{\cos\frac{\pi x}{2}} = \left|\begin{array}{l}\text{Transformacija}\\ \text{će uslijediti}\\ \text{nakon smjene}\end{array}\right| = \lim_{x\to 1}\frac{(1-x)}{\cos\frac{\pi x}{2}} = \left|\begin{array}{l}\text{smjena: } 1-x = t\\ x = 1-t,\ t\to 0\end{array}\right| =$$

$$= \lim_{x\to 1}\frac{t}{\cos\frac{\pi}{2}(1-t)} = \lim_{x\to 1}\frac{t}{\cos\left(\frac{\pi}{2}-\frac{\pi t}{2}\right)} = \left|\begin{array}{l}\text{ili transformacije:}\\ \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ \cos(\pi/2-\alpha) = \sin\alpha\end{array}\right| =$$

$$= \lim_{x\to 1}\frac{t}{\sin\frac{\pi t}{2}} = \left|\lim_{x\to 1}\frac{\left(\sin\frac{\pi t}{2}\right)^{-1}\cdot\left(\frac{\pi t}{2}\right)}{\frac{\pi t}{2}}\right\}\text{OVO NIKAD!}\right| =$$

$$= \lim_{x\to 1}\underbrace{\frac{\frac{\pi}{2}\cdot t}{\sin\frac{\pi t}{2}}}_{\to 1}\cdot\frac{1}{\frac{\pi}{2}} = \underline{\frac{2}{\pi}}$$

(Z) $L=\lim\limits_{x\to+\infty}\frac{x^2}{e^x}=\lim\limits_{x\to+\infty}\underbrace{\frac{x^2}{(x+1)^2}}_{1}\cdot\underbrace{\frac{(x+1)^2}{e^{x+1}}}_{L}\cdot e = L e \wedge 0<\frac{x^2}{e^x}<1$.

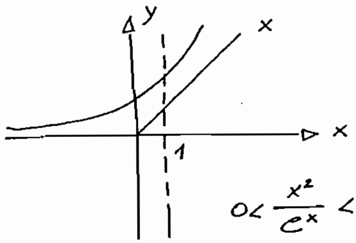
$L=L\cdot e \Rightarrow 0=L\cdot e-L \Rightarrow L(e-1)=0$

Predpostavka:

1. $L=\text{const.}\neq 0$ [ne] jer je $e>1$
2. $L=+\infty$ [ne]
3. $L=0$ (da)

* $L=\left(\lim\limits_{x\to\infty}\frac{x}{e^{x/2}}\right)^2$

$e^{x/2}>x,\ x\gg$



$0<\frac{x^2}{e^x}<1$

(Z) Riješiti limes: $L=\lim\limits_{x\to+\infty}\frac{\ln^2 x}{x}$!

$\lim\limits_{x\to+\infty}\frac{\ln^2 x}{x}\overset{\frac{\infty}{\infty}}{=}\left|\begin{matrix}\text{smjena:}\\ \ln x=t\Rightarrow x=e^t,\ t\to+\infty\end{matrix}\right|=\lim\limits_{t\to+\infty}\frac{t^2}{e^t}=\left|\begin{matrix}t=x\\ x\to+\infty\end{matrix}\right|=$

$=\lim\limits_{x\to+\infty}\frac{x^2}{e^x}=0$

* koristili smo formulu: $\lim\limits_{x\to+\infty}\frac{x^\alpha}{e^{\beta x}}=0$
$\forall\alpha,\ \beta>0$

kod nas je bilo: $\alpha=2$
$\beta=1>0$.

* Potrebno je poznavati sljedeće transformacije:

$\sin(\alpha\pm\beta)=\sin\alpha\cdot\cos\beta\pm\cos\alpha\cdot\sin\beta$
$\cos(\alpha\pm\beta)=\cos\alpha\cdot\cos\beta\mp\sin\alpha\sin\beta$
$\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$
$\sin\alpha-\sin\beta=2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha-\sin^2\alpha$$

$$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\cos\alpha = \cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}$$

$$1+\cos\alpha = 2\cos^2\frac{\alpha}{2}$$

$$1-\cos\alpha = 2\sin^2\frac{\alpha}{2}$$

(Z) $L=\lim\limits_{x\to 0}\dfrac{\cos xe^{x}-\cos xe^{-x}}{x^3}$

$$\lim_{x\to 0}\frac{\cos xe^{x}-\cos xe^{-x}}{x^3}\overset{\frac{0}{0}}{=}\left|\cos\alpha-\cos\beta=-2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right|=$$

$$=\lim_{x\to 0}\frac{1}{x^3}\left(-2\cdot\sin\left(\frac{xe^{x}+xe^{-x}}{2}\right)\cdot\sin\left(\frac{xe^{x}-xe^{-x}}{2}\right)\right)=$$

$$=-2\cdot\lim_{x\to 0}\frac{1}{x^3}\sin\frac{x(e^{x}+e^{-x})}{2}\sin\frac{x(e^{x}-e^{-x})}{2}\overset{\frac{0}{0}}{=}\left|\begin{matrix}\sin u\sim u,\\ u\to 0.\end{matrix}\right|=$$

$$=-2\cdot\lim_{x\to 0}\frac{1}{x^3}\frac{x(e^{x}+e^{-x})}{2}\cdot\frac{x(e^{x}-e^{-x})}{2}=-2\lim_{x\to 0}\frac{x^2(e^{2x}-e^{-2x})}{4x^3}=$$

$$=-2\lim_{x\to 0}\frac{e^{-2x}(e^{4x}-1)}{4x}=\left|\lim_{x\to 0}\frac{e^{x}-1}{x}=1\right|=-2\lim_{x\to 0}e^{-2x}=\underline{-2}.$$

(Z) $L=\lim\limits_{x\to\infty}x^2\cdot\ln\dfrac{\frac{1}{x}}{\sin\frac{1}{x}}$

$$\lim_{x\to\infty}x^2\cdot\ln\frac{\frac{1}{x}}{\sin\frac{1}{x}}\overset{\infty\cdot 0}{=}\left|\begin{matrix}\frac{1}{x}=t, & t\to 0\\ x=t, & x\to 0\end{matrix}\right|=\lim_{x\to 0}\frac{1}{x^2}\cdot\ln\frac{x}{\sin x}=$$

$$=\lim_{x\to 0}\frac{1}{x^2}(\ln x-\ln\sin x)=\lim_{x\to 0}\frac{1}{x^2}(-\ln\sin x+\ln x)=\lim_{x\to 0}\frac{1}{x^2}\cdot(-(\ln\sin x-\ln x))=$$

$$=-\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x}=\left|\begin{matrix}\ln u\sim u-1,\\ u\to 1\end{matrix}\right|=-\lim_{x\to 0}\frac{1}{x^2}\left(\frac{\sin x}{x}-1\right)=$$

$$=-\lim_{x\to 0}\frac{\sin x-x}{x^3}=\underline{-\frac{1}{6}}.$$

(Z) $L=\lim\limits_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}}$

$\lim\limits_{x\to 0}\left(\frac{\arcsin x}{x}\right)^{\frac{1}{x^2}} \overset{1^\infty}{=} L \,/\cdot\ln \Rightarrow \ln L = \lim\limits_{x\to 0}\frac{1}{x^2}\cdot\ln\left(\frac{\arcsin x}{x}\right) \overset{0\cdot\infty}{=}$

$= \left|\begin{matrix}\ln u \sim u-1,\\ u\to 1\end{matrix}\right| = \lim\limits_{x\to 0}\frac{1}{x^2}\left(\frac{\arcsin x}{x}-1\right) = \lim\limits_{x\to 0}\frac{1}{x^2}\left(\frac{\arcsin x - x}{x}\right) =$

$= \left|\begin{matrix}\arcsin x = t\\ x=\sin t\ ,\ t\to 0\end{matrix}\right| = \lim\limits_{t\to 0}\frac{t-\sin t}{(\sin t)^3} = \left|\begin{matrix}\sin u \sim u,\\ u\to 0\end{matrix}\right| = \lim\limits_{t\to 0}\frac{t-\sin t}{t^3} =$

$= -\lim\limits_{x\to 0}\underbrace{\frac{\sin x - x}{x^3}}_{\to \frac{1}{6}} = \underline{-\frac{1}{6}}.$

$\ln L = -\frac{1}{6} \Rightarrow \underline{L=\frac{1}{\sqrt[6]{e}}}.$

(Z) $L=\lim\limits_{x\to 0}\frac{\cos x\cos 2x\cdot\ldots\cdot\cos nx - 1}{x^2} \overset{\frac{0}{0}}{=} \left|\begin{matrix}A^\alpha = e^{\alpha\ln A},\\ A>0\end{matrix}\right| =$

$= \lim\limits_{x\to 0}\frac{e^{\ln\prod\limits_1^n \cos kx}}{x^2} = \lim\limits_{x\to 0}\frac{e^{\sum\limits_1^n \ln\cos kx}-1}{x^2} = \left|\begin{matrix}e^u-1\sim u,\\ u\to 0\end{matrix}\right| =$

$= \lim\limits_{x\to 0}\frac{\sum\limits_1^n \ln\cos kx}{x^2} = \sum\limits_1^n\underbrace{\lim\limits_{x\to 0}\frac{\ln\cos kx}{x^2}}_{L_k} \Rightarrow$

* Limes sume postoji ako bar jedan od pod sumnih limesa postoji.

$\Rightarrow L_k = \lim\limits_{x\to 0}\frac{\ln\cos kx}{x^2} \overset{\frac{0}{0}}{=} \left|\begin{matrix}\ln u\sim u-1,\\ u\to 1\end{matrix}\right| = \lim\limits_{x\to 0}\frac{\cos kx - 1}{x^2} \overset{\frac{0}{0}}{=}$

$= \left|\cos\alpha - 1 = -2\sin^2\frac{\alpha}{2}\right| = \lim\limits_{x\to 0}(-2)\underbrace{\frac{\sin^2\frac{kx}{2}}{\frac{x^2}{4}\cdot k^2}}_{\to 1}\cdot\frac{k^2}{4} = \underline{-\frac{k^2}{2}}.$

$\Rightarrow$ da postoji limes sume!

$L=\lim\limits_{x\to 0}\sum\limits_1^n -\frac{k^2}{2} = \sum\limits_1^n -\frac{k^2}{2} = -\frac{1}{2}\sum\limits_1^n k^2 = -\frac{1}{2}(1^2+2^2+\ldots+n^2) =$

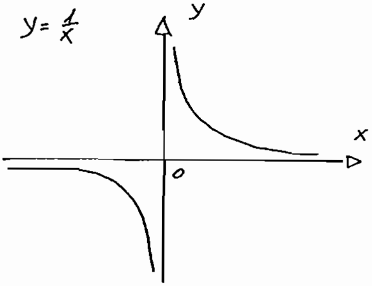
$= \underline{-\frac{1}{2}\cdot\frac{n}{6}(n+1)(2n+1)}.$

* Naći lijevi i desni limes f-je:

$$f(x) = \frac{e^{\frac{1}{x}} + a^2 + a}{e^{1/x} + 2}, \quad x \neq 0,$$ u tački $x = 0$. Za koje vrijednosti realnog parametra a limes f-je postoji?

*

$y = \frac{1}{x}$

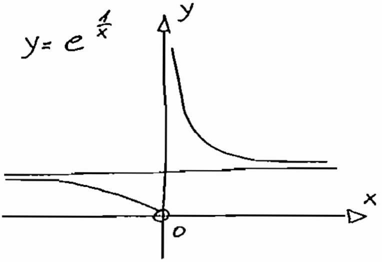


$xy = 1$
$xy - 1 = 0$

$A = 0$, $2B = 1$, $C = 0$, $2D = 0$, $2E = 0$, $F = -1$ } karakteristike opšte jednačine od $xy - 1 = 0$.

$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$

$y = e^{\frac{1}{x}}$



$$\lim_{x \to 0_-} e^{\frac{1}{x}} = e^{\frac{1}{0_-}} = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = \underline{0_+}.$$

$$\lim_{x \to 0_+} e^{\frac{1}{x}} = e^{\frac{1}{0}} = e^{\infty} = \underline{+\infty}.$$

$$L_1 = \lim_{x \to 0_-} \frac{e^{\frac{1}{x}} + a^2 + a}{e^{\frac{1}{x}} + 2} = \underline{\frac{a^2 + a}{2}}..$$

$$L_2 = \lim_{x \to 0_+} \frac{e^{\frac{1}{x}} + a^2 + a}{e^{\frac{1}{x}} + 2} \overset{\frac{\infty}{\infty}}{=} \left| \begin{matrix} \text{dijelim izroz sa:} \\ e^{\frac{1}{x}},\ x \neq 0 \end{matrix} \right| = \lim_{x \to 0_+} \frac{1 + \frac{a^2 + a}{e^{1/x}}}{1 + \frac{2}{e^{1/x}}} = \underline{1}..$$

$L_1 = \frac{a^2 + a}{2}$
$L_2 = 1$ } (1) $L_1 \neq L_2$
(2) $L_1 = L_2 = L \Rightarrow$ to se može desiti u slučaju kada je $\frac{a^2 + a}{2} = 1 \Rightarrow a^2 + a - 2 = 0$

$$a_{1,2} = \frac{-1 \pm 3}{2} \Rightarrow \begin{matrix} a_1 = -2 \\ a_2 = 1 \end{matrix}$$

pa za $a_1 = -2$ i $a_1 = 1$ f-ja ima KONAČNU GRANIČNU VRIJEDNOST.

(2) Odrediti konstante a i b tako da je:

$$\lim_{x\to\infty}\left(ax+b-\frac{x^3-1}{x^2+1}\right)=0.$$

Odrediti realni parametar λ tako da

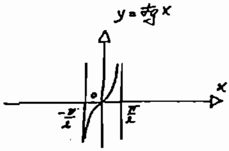
$$\lim_{x\to \pi/2}\left(\operatorname{tg}x+\lambda\operatorname{tg}3x\right)=const.,$$ i naći const.!

$$L=\lim_{x\to+\infty}\left(ax+b-\frac{x^3-1}{x^2+1}\right)=\lim_{x\to+\infty}\left(\frac{ax(x^2+1)+b(x^2+1)-(x^3-1)}{\underbrace{\frac{x^2+1}{x^2}}_{\to 1}\cdot x^2}\right)=$$

$$=\lim_{x\to+\infty}\frac{ax^3+ax+bx^2+b-x^3-1}{x^2}=\lim_{x\to+\infty}\frac{x^3(a-1)+bx^2+ax+b-1}{x^2}=$$

$$=\lim_{x\to+\infty}\frac{(a-1)x^3+bx^2+ax+(b-1)}{x^2}=0$$

Za $a=1 \Rightarrow \lim_{x\to\infty}\left(b+\frac{a}{x}+\frac{b-1}{x^2}\right)\Rightarrow b=0.$



$\underline{L=0 \text{ za } a=1 \wedge b=0}$.

$$L=\lim_{x\to\pi/2}\left(\operatorname{tg}x+\lambda\operatorname{tg}3x\right)$$

$$* \quad \operatorname{tg}(\alpha\pm\beta)=\frac{\operatorname{tg}\alpha\pm\operatorname{tg}\beta}{1\mp\operatorname{tg}\alpha\cdot\operatorname{tg}\beta}\quad ; \quad \operatorname{ctg}(\alpha\pm\beta)=\frac{\operatorname{ctg}\alpha\cdot\operatorname{ctg}\beta\mp 1}{\operatorname{ctg}\beta\pm\operatorname{ctg}\alpha}$$

$$\operatorname{tg}2\alpha=\frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}.$$

$$L=\lim_{x\to\pi/2}\left(\operatorname{tg}x+\lambda\operatorname{tg}3x\right)=\left|\begin{array}{l}\operatorname{tg}x=t,\ t\to+\infty\\ \operatorname{tg}3x=\operatorname{tg}(x+2x)=\dfrac{\operatorname{tg}x+\operatorname{tg}2x}{1-\operatorname{tg}x\cdot\operatorname{tg}2x}=\end{array}\right.$$

$$=\frac{t+\frac{2t}{1-t^2}}{1-\frac{2t^2}{1-t^2}}=\left.\frac{3t-t^3}{1-3t^2}\right|=\left|\begin{array}{l}x\equiv t,\\ x\to+\infty\end{array}\right|=\lim_{x\to+\infty}\left(x+\lambda\cdot\frac{x^3-3x}{3x^2-1}\right)=$$

$$=\lim_{x\to+\infty}\frac{3x^3-x+\lambda x^3-3\lambda x}{3x^2}=\lim_{x\to+\infty}\left[(3+\lambda)x-\frac{1+3\lambda}{x}\right]=0\Rightarrow$$

$$\Rightarrow \underline{\lambda=-3}.$$

② koristeči ekvivalencije beskonačno malih veličina: $f_1(x) = \sqrt[n]{1+x} - 1$, $f_2(x) = \frac{x}{n}$, $n \in N$, $x \to 0$ ili za x po volji malo, odrediti približno $\sqrt{912}$, $\sqrt[3]{1042}$, $\sqrt[4]{10124}$!

* Najprije treba pokazati da su $f_1(x)$ i $f_2(x)$ beskonačno male veličine $x \to 0$.

$$f_1(x) = \sqrt[n]{1+x} - 1 = (1+x)^{\frac{1}{n}} - 1 = e^{\frac{1}{n}\ln(1+x)} - 1 = \left| \begin{array}{c} e^u - 1 \sim u, \\ u \to 0,\ x \to 0 \end{array} \right| =$$

$$= \frac{1}{n}\ln(1+x) = \left| \begin{array}{c} \ln(1+x) \sim x, \\ x \to 0 \end{array} \right| = \frac{1}{n} \cdot x \xrightarrow[x \to 0]{} $$

$$f_2(x) = \frac{x}{n} \xrightarrow{x \to 0} 0$$

Sada smo pokazali da je $f_1(x) \sim f_2(x)$ kad $x \to 0$ ili

$$\lim_{x \to 0} \frac{f_1(x)}{f_2(x)} \overset{\frac{0}{0}}{=} 1$$

* za $|x| <<$

$$\sqrt[n]{1+x} - 1 = \frac{x}{n}$$

$$\sqrt[n]{1+x} = 1 + \frac{x}{n}$$

→ red korijena

$$\sqrt{912} = \sqrt{30^2 + 12} = 30\sqrt{1 + \frac{12}{30^2}} = 30\sqrt{1 + \frac{12}{900}} = 30 \cdot \left(1 + \frac{\frac{12}{900}}{2}\right) =$$

↓ red korijena

$$= 30\left(1 + \frac{12}{2 \cdot 900}\right) = 30{,}2$$

$$\sqrt[3]{1042} = \sqrt[3]{10^3 + 42} = 10\sqrt[3]{1 + \frac{42}{1000}} = 10\left(1 + \frac{42}{3000}\right)$$

$$\sqrt[4]{10124} = \sqrt[4]{10^4 + 124} = 10\sqrt[4]{1 + \frac{124}{10^4}} = 10\left(1 + \frac{124}{4 \cdot 10000}\right) = \underline{10{,}0031}$$

* koristimo formulu:

$$\boxed{\sqrt[n]{1+x} = 1 + \frac{x}{n}} \quad \text{za } |x| <<$$

ZADACI IZ LJAŠKA:

(2) Odrediti red BMV. (beskonačno male veličine)

$f(x) = \sqrt{1-2x} - \sqrt[3]{1-3x}\,, \quad x\to 0$

Zadatak se svodi na traženje realnog parametra $\alpha>0$ tako da je: $\lim\limits_{x\to 0_+} \frac{f(x)}{x^\alpha} \overset{\frac{0}{0}}{=} const. \neq 0.$

$$L = \lim_{x\to 0_+} \frac{\sqrt{1-2x} - \sqrt[3]{1-3x}}{x^\alpha} = \left| \begin{array}{l} \sqrt{1-2x} = \sqrt[3]{(\sqrt{1-2x})^3}; \\ \sqrt[3]{a} - \sqrt[3]{b} = \frac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}} \\ a>0, b>0 \end{array} \right| =$$

$$= \left| \begin{array}{l} \text{kod nas je:} \\ a = (\sqrt{1-2x})^3 \\ b = 1-3x \end{array} \right| = \lim_{x\to 0_+} \frac{\sqrt{(1-2x)^3} - (1-3x)}{x^\alpha \underbrace{\left[(1-2x)^3 + \sqrt[3]{(1-3x)(\sqrt{1-2x})^3} + \sqrt[3]{(1-3x)^2}\right]}_{\to 3}} =$$

$$= |\text{racionalizacija}| = \lim_{x\to 0_+} \frac{(1-2x)^3 - (1-3x)^2}{3x^\alpha \underbrace{\left((\sqrt{1-2x})^3 + (1-3x)\right)}_{\to 2}} =$$

$$= \lim_{x\to 0_+} \frac{1-6x+12x^2-8x^3-(1-6x+9x^2)}{x^\alpha \cdot 6} = \frac{1}{6}\lim_{x\to 0_+} \frac{x^2\overbrace{(3-8x)}^{\to 3}}{x^\alpha} =$$

$$= \frac{1}{2}\lim_{x\to 0_+} x^{2-\alpha} = \frac{1}{2} \quad \Big|_{\alpha=2}$$

Uradimo sada zadatak za $\alpha = 2$.

$$\lim_{x\to 0_+} \frac{\sqrt{1-2x} - \sqrt[3]{1-3x}}{x^2} = \frac{1}{2} = \lim_{x\to 0_-} \frac{\sqrt{1-2x} - \sqrt[3]{1-3x}}{x^2} =$$

$$= \lim_{x\to 0} \frac{\sqrt{1-2x} - \sqrt[3]{1-3x}}{x^2}$$

II) ekvivalencija:

$$(1+x)^\alpha \sim 1+\alpha x + \frac{\alpha(\alpha-1)x^2}{2}, \quad x\to 0, \ \forall \alpha \ \text{ili}$$

$$(1+x)^\alpha = 1+\alpha x + \frac{\alpha(\alpha-1)x^2}{2} + O(x^3), \ x\to 0$$

$$f(x) = (1-2x)^{\frac{1}{2}} - (1-3x)^{\frac{1}{3}} = 1 - 2x\cdot\frac{1}{2} + \frac{\frac{1}{2}(-\frac{1}{2})}{2}\cdot 4x^2 + O_1(x^3) -$$

$$-\left(1 - 3x\cdot\frac{1}{3} + \frac{\frac{1}{3}(-\frac{2}{3})}{2}\cdot 9x^2 + O_2(x^3)\right) =$$

$$= x^2\left(1-\frac{1}{2}\right) + O(x^3) = \frac{1}{2}x^2 + O(x^3) \sim \frac{1}{2}x^2.$$

(Z) Odrediti red BMV. $f(x) = \operatorname{tg} x - \sin x,\ x \to 0$!

$$f(x) = \frac{\sin x}{\cos x} - \sin x = \sin x\left(\frac{1-\cos x}{\cos x}\right) = \frac{\sin x}{\cos x}(1-\cos x) =$$

$$= \underbrace{\frac{\sin x}{\cos x}}_{\to 1} \cdot 2\sin^2\frac{x}{2} \cong x \cdot 2 \cdot \frac{x^2}{4} = \frac{1}{2}x^3$$

$$\lim_{x\to 0}\frac{f(x)}{x^3} = \frac{1}{2}, \quad \underline{\alpha = 3}. \text{ - red BMV}$$

* Ovdje smo transformisali izraz i kada smo ga doveli do $\frac{\sin x}{\cos x} \cdot 2\sin^2\frac{x}{2}$ primijenili smo smjenu:

$\left|\begin{matrix}\sin u \sim u, \\ u \to 0\end{matrix}\right|$, a $\cos x \to 1,\ x \to 0$. Tek nakon toga tj. nakon dobijenog rezultata $\frac{1}{2}x^3$ primijenili smo formulu za traženje reda BMV. $\lim\limits_{x\to 0}\frac{f(x)}{x^\alpha} = \text{const.} \neq 0$.
Da bi $\frac{1}{2}x^3$ bio const. $\neq 0$ treba ga dijeliti sa x^3 pa je $\alpha = 3$ red BMV.!

(Z) $$\lim_{x\to 0}\frac{\cos(a+2x) - 2\cos(a+x) + \cos a}{x^2}$$

$$\lim_{x\to 0}\frac{\cos(a+2x) - 2\cos(a+x) + \cos a}{x^2} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{[\cos(a+2x) - \cos(a+x)]}{x^2} -$$

$$- \lim_{x\to 0}\frac{[\cos(a+x) - \cos a]}{x^2} = \lim_{x\to 0}\frac{-2\sin\left(a+\frac{3x}{2}\right)\sin\frac{x}{2} + 2\sin\left(a+\frac{x}{2}\right)\sin\frac{x}{2}}{x^2} =$$

$$= \lim_{x\to 0}\underbrace{\frac{\sin\frac{x}{2}}{\frac{x}{2}}}_{\to 1} \cdot \frac{\sin\left(a+\frac{x}{2}\right) - \sin\left(a+\frac{3x}{2}\right)}{x} = 2\lim_{x\to 0}\frac{\sin\left(-\frac{x}{2}\right)\cos(a+x)}{x} =$$

$$= 2\lim_{x\to 0}\frac{-\frac{x}{2}\cos a}{x} = \underline{-\cos a}.$$

* Ako imamo kakav eksponent u izrazu idemo preko broja e, a treba kod korijena probati i racionalizaciju.

(Z) $$\lim_{x\to\infty}\left(\frac{x-2}{2x+1}\right)^{x^2} \overset{1^\infty}{=} \lim_{x\to\infty}\left(\frac{1-\frac{2}{x}}{2+\frac{1}{x}}\right)^{x^2} = \lim_{x\to\infty}\left(\frac{1}{2}\right)^{x^2} = \lim_{x\to\infty}\frac{1}{2^{x^2}} = \underline{0}.$$

Ⓩ $\lim\limits_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2} \overset{1^\infty}{=} L \quad /\ominus\ln$

$\ln L = \lim\limits_{x\to\infty} x^2\cdot\ln\left(\frac{x^2+1}{x^2-2}\right) \overset{\infty\cdot 0}{=} \left|\begin{matrix}\ln u \sim u-1,\\ u\to 1\end{matrix}\right| = \lim\limits_{x\to\infty} x^2\left(\frac{x^2+1}{x^2-2}-1\right) =$

$= \lim\limits_{x\to\infty} \frac{3x^2}{x^2-2} = \lim\limits_{x\to\infty}\frac{3}{1-\underbrace{\frac{2}{x^2}}_{\to 0}} = \underline{3}.$

$\ln L = 3 \Rightarrow \underline{L = e^3}.$

Ⓩ $\lim\limits_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx}$

$\lim\limits_{x\to 0}\frac{\ln\cos ax}{\ln\cos bx} \overset{\frac{0}{0}}{=} \left|\begin{matrix}\ln u \sim u-1,\\ u\to 1\end{matrix}\right| = \lim\limits_{x\to 0}\frac{\cos ax-1}{\cos bx-1} \overset{\frac{0}{0}}{=}$

$= \lim\limits_{x\to 0}\frac{\sin^2\frac{ax}{2}}{\sin^2\frac{bx}{2}} = \left|\begin{matrix}\sin u\sim u,\\ u\to 0\end{matrix}\right| = \lim\limits_{x\to 0}\frac{a^2x^2}{4\cdot\frac{b^2x^2}{4}} = \frac{a^2}{b^2} = \underline{\left(\frac{a}{b}\right)^2}.$

Ⓩ $L = \lim\limits_{x\to 0}\frac{1-\cos^\nu x}{x^2}, \ \nu\neq 0$

$\lim\limits_{x\to 0}\frac{1-\cos^\nu x}{x^2} \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\frac{-(\cos^\nu x-1)}{x^2} = \lim\limits_{x\to 0}\frac{-(e^{\nu\ln\cos x}-1)}{x^2} =$

$= \left|\begin{matrix}e^u-1\sim u;\\ u\to 0\end{matrix}\right| = \lim\limits_{x\to 0}\frac{+(e^{\nu\ln\cos x}-1)}{-x^2} = \lim\limits_{x\to 0}\frac{\nu\ln\cos x}{-x^2} \overset{\frac{0}{0}}{=} \left|\begin{matrix}\ln u\sim u-1,\\ u\to 1\end{matrix}\right| =$

$= \lim\limits_{x\to 0}\frac{\nu(\cos x-1)}{-x^2} = \lim\limits_{x\to 0}\frac{\nu\cdot 2\cdot\sin^2\frac{x}{2}}{+x^2} = \lim\limits_{x\to 0} 4\cdot\frac{2\nu\sin^2\frac{x}{2}}{\frac{x^2}{4}} =$

$= \lim\limits_{x\to 0}\frac{2\nu}{4} = \underline{\frac{\nu}{2}}.$

Ⓩ $L = \lim\limits_{x\to 0}\frac{\mathrm{sh}^2 x}{\ln(\mathrm{ch}\,3x)} \overset{0/0}{=} L$

$\lim\limits_{x\to 0}\frac{\mathrm{sh}^2 x}{\ln(\mathrm{ch}\,3x)} = \left|\begin{matrix}\mathrm{sh}\,u\sim u,\ u\to 0\\ \ln u\sim u-1,\\ u\to 1\end{matrix}\right| = \lim\limits_{x\to 0}\frac{x^2}{\mathrm{ch}\,3x-1} \overset{\frac{0}{0}}{=} \Big|$

$= \left|\mathrm{ch}\,\alpha-1 = 2\,\mathrm{sh}^2\frac{\alpha}{2}\right| = \lim\limits_{x\to 0}\frac{x^2}{2\,\mathrm{sh}^2\frac{3x}{2}} \overset{\frac{0}{0}}{=} \left|\begin{matrix}\mathrm{sh}\,u\sim u,\\ u\to 0\end{matrix}\right| = \frac{1}{2}\lim\limits_{x\to 0}\frac{4x^2}{9x^2} = \underline{\frac{2}{9}}.$

(Z) $L=\lim_{x\to\infty} \ln(1+2^x)\cdot\ln\left(1+\frac{3}{x}\right) \overset{\infty\cdot 0}{=} \left| \begin{matrix}\ln(1+u)\sim u,\\ u\to 0\end{matrix} \right| = \lim_{x\to+\infty}\left[\ln(1+2^x)\right]\cdot\frac{3}{x} =$

$= \lim_{x\to+\infty}\left[\ln 2^x\left(1+\frac{1}{2^x}\right)\right]\cdot\frac{3}{x} = 3\cdot\lim_{x\to+\infty}\frac{x\ln 2+\overbrace{\ln\left(1+\frac{1}{2^x}\right)}^{\to 0}}{x} = 3\lim_{x\to\infty}\frac{x\ln 2}{x} =$

$= \ln 8 = \underline{3\ln 2}.$

$\underline{L = 3\ln 2 = \ln 8}.$

(Z) $\lim_{n\to\infty}\sin\pi\sqrt{n^2+1} = \lim_{n\to\infty}\sin\left(\pi\sqrt{n^2+1}-n\pi+n\pi\right)=$

$= \lim_{n\to\infty}\sin\left[\left(\pi\sqrt{n^2+1}-n\pi\right)+(n\pi)\right] = \lim_{n\to\infty}\sin\left[\underbrace{\left(\pi\sqrt{n^2+1}-n\pi\right)}_{\alpha}+\underbrace{(n\pi)}_{\beta}\right] =$

$= \lim_{n\to\infty}\cos n\pi\,\sin\pi\left(\sqrt{n^2+1}-n\right) = \lim_{n\to\infty}\underbrace{(-1)^n}\cdot\sin\underbrace{\frac{\pi}{\sqrt{n^2+1}+n}}_{\text{BMV}} = \underline{0}.$

↳ ograničena veličina, umjesto $\cos n\pi$ možemo pisati $(-1)^n$.

(Z) $\lim_{n\to\infty}\prod_{k=1}^{n}\left(1+\frac{k}{n^2}\right)$

$\frac{1}{n^2}\le\frac{k}{n^2}\le\frac{n}{n^2}\ /\circ\lim_{n\to\infty} \Rightarrow \lim_{n\to\infty}\frac{k}{n^2}=0,\ \forall k=1,2,\dots,n$

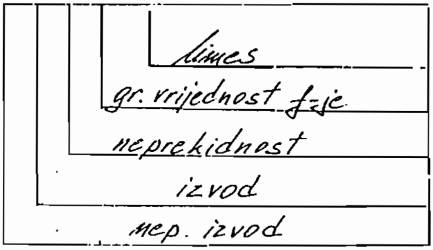
$\frac{k}{n^2}$ ravnomjerno konvergira $\to 0$ i pišemo $\frac{k}{n^2}\xrightarrow{n\to\infty} 0$

$L=\lim_{n\to\infty} e^{\ln\prod_{1}^{n}\left(1+\frac{k}{n^2}\right)} = \lim_{n\to\infty} e^{\sum_{1}^{n}\ln\left(1+\frac{k}{n^2}\right)} = \left| \begin{matrix}\ln(1+u)\sim u,\\ u\to 0, \text{ jer}\\ \frac{k}{n^2} \text{ ravnomjerno}\\ \text{konvergira}\end{matrix} \right| =$

$= \lim_{n\to\infty} e^{\sum_{k=1}^{n}\frac{k}{n^2}} = e^{\lim_{n\to\infty}\frac{1+2+\dots+n}{n^2}} = e^{1/2} = \sqrt{e}$

* $1+2+3+\dots+n = \frac{n}{2}(n+1) = \frac{n^2}{2}+\frac{n}{2} = \frac{n^2+n}{2}$

PRIMJENA GRANIČNIH PROCESA

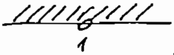


(Z) Data je sljedeća f-ja: $f(x)=\begin{cases} x^2+1, & x<1 \\ \dfrac{x^2-1}{x-1}, & x>1 \end{cases}$

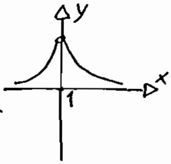
a) pokazati da data f-ja ima konačnu graničnu vrijednost u $x=1$

b) definisati f-ju neprekidnu za svako x.

$$f(x)=\begin{cases} x^2+1, & x<1 \\ \dfrac{x^2-1}{x-1}, & x>1 \end{cases} \qquad * \quad x=1$$



$\Rightarrow$



$$L_1=\lim_{x\to 1\uparrow} f(x)=\lim_{x\to 1\uparrow}(x^2+1)=2$$

$$L_2=\lim_{x\to 1\downarrow} f(x)=\lim_{x\to 1\downarrow}\frac{x^2-1}{x-1}\overset{\frac{0}{0}}{=}\lim_{x\to 1\downarrow}\frac{(x-1)(x+1)}{x-1}=2$$

$$L_1=L_2=L=\lim_{x\to 1} f(x)=2$$

b)

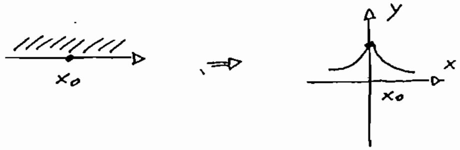
$$F(x)=\begin{cases} f(x), & x\neq 1 \\ \lim\limits_{x\to 1} f(x), & x=1 \end{cases} \qquad F(x)=\begin{cases} x^2+1, & x<1 \\ 2, & x=1 \\ x+1, & x>1 \end{cases}$$

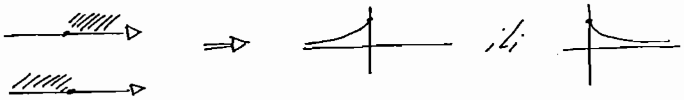
Ovo je tzv. produženje po neprekidnosti. O čemu se radi? F-ja ima limes u tački $x=1$ međutim nije definisana u toj tački. Zato definišemo novu f-ju dodavanjem „$x=1$ i u tom slučaju je $f(x)=2$" pa imamo novu f-ju $F(x)$.

* Neprekidnost u tački i intervalu

Definicija: F-ja $f(x)$ na E_x je neprekidna u tč. $x = x_0$ ako je cjelokupna okolina $x_0 \in E_x$, $O(x_0) \in E_x$ i vrijedi: $\lim\limits_{x \to x_0} f(x) = f(x_0)$

Ako je narušen uslov $O(x_0) \in E_x$ tada se radi o neprekidnosti u gr. tačkama intervala definisanosti:





Tačke moraju biti pune jer idemo u konačnost!

Definicija: $\lim\limits_{x \to x_0} f(x) = f(x_0) \sim \Delta y = |f(x) - f(x_0)| < \varepsilon$

za $|x - x_0| < \delta(\varepsilon)$

$\delta(\varepsilon) \xrightarrow{\varepsilon \to 0_+} 0$

Ⓩ Odrediti α i β tako da je f-ja data sa:

$$f(x) = \begin{cases} \alpha x^2 + 1, & x < 2 \\ 5, & x = 2 \\ x + \beta, & 2 < x \end{cases}$$ neprekidna u tački $x = 2$.

$$L_1 = \lim_{x \to 2\uparrow} f(x) = \lim_{x \to 2\uparrow} \alpha x^2 + 1 = 4\alpha + 1$$
$$L_2 = \lim_{x \to 2\downarrow} f(x) = \lim_{x \to 2\downarrow} (x + \beta) = \beta + 2$$
$$\Rightarrow L_1 = L_2 = L = f(2)$$
$$4\alpha + 1 = \beta + 2 = 5$$

$\alpha = 1$

$\beta = 3$

$$f(x) = \begin{cases} x^2 + 1, & x < 2 \\ 5, & x = 2 \\ x + 3, & x > 2 \end{cases}$$

Ova f-ja je neprekidna u tački $x = 2$!

ⓩ Odrediti $\lim\limits_{x\to1}\frac{f(x+2)-f(2x+1)}{x^2-1}=L$ ako je $f\left(\frac{x}{x+2}\right)=x$,

, $x\neq-2$.

$$L\overset{\frac{0}{0}}{=}\lim_{x\to1}\frac{f(x+2)-f(2x+1)}{x^2-1}$$

$$\frac{x}{x+2}=u \Rightarrow x=ux+2u \Rightarrow x=\frac{2u}{1-u}. \Rightarrow \frac{2u}{1-u}\neq-2.$$

$$f(u)=\frac{2u}{1-u},\quad u\neq1$$

1° $u\equiv x+2$ $\quad f(x+2)=\frac{2x+4}{1-x-2}=\frac{2x+4}{-x-1},\quad x\neq-1$

2° $u\equiv 2x+1$ $\quad f(2x+1)=\frac{4x+2}{-2x},\quad x\neq0$

$$L\overset{\frac{0}{0}}{=}\lim_{x\to1}\frac{-\frac{2x+4}{x+1}+\frac{1+2x}{x}}{(x-1)\cdot(x+1)}=\lim_{x\to1}\frac{-2x^2-4x+1+2x+x+2x^2}{2(x-1)x(x+1)}=$$

$$=\lim_{x\to1}\frac{-(x-1)}{4(x-1)}=-\frac{1}{4}.$$

ⓩ Ispitati neprekidnost f-je date sa:

$$f(x)=\begin{cases}x^n\sin\frac{1}{x}, & x\neq0\\ 0, & x=0\end{cases},\quad n=0,1,2,\ldots$$

1) $\forall x\neq0$ data fja je neprekidna jer je sastoji od elementarnih f-ja x^n; $\sin\frac{1}{x}$.

Treba ispitati neprekidnost date f-je u $x=0$.

2) $n=0$, $f(x)=\begin{cases}\sin\frac{1}{x}, & x\neq0\\ 0, & x=0\end{cases}$

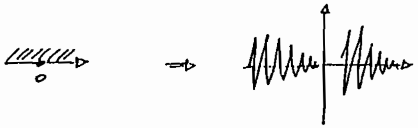
$f(0)=0$

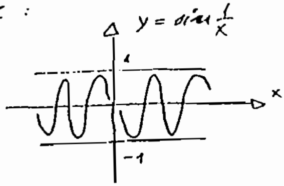
$$L_2=\lim_{x\to0_+}\sin\frac{1}{x}=\left|\begin{array}{l}x=\frac{1}{2n\pi+\alpha},\ \forall\alpha\in\mathbb{R}\\ n\to+\infty\end{array}\right|=\lim_{n\to+\infty}\sin(2n\pi+\alpha)=$$

$$=\sin\alpha\begin{cases}0, & \alpha=0\\ 1, & \alpha=\frac{\pi}{2}\end{cases}\quad \text{divergira u širem smislu.}$$

L_1 takođe OŠS.

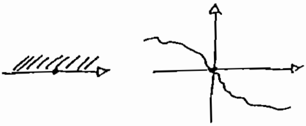
za $n=0$ imamo prekid II vrste :





3) $n=1$, $f(0)=0$, $f(x) = x \sin\frac{1}{x}$

$\lim\limits_{x\to 0} x \sin\frac{1}{x} = 0 = f(0) \Rightarrow n=1$



(Z) Ispitati neprekidnost f-je date sa

$$f(x) = \lim_{n\to\infty} (x \operatorname{arctg} nx)$$

a) $x=0 \Rightarrow$ aksiom prioriteta $f(0)=0$

b) $x>0$ $\operatorname{arctg} nx \xrightarrow{n\to\infty} \pi/2$

c) $x<0$ $\operatorname{arctg} nx \xrightarrow{n\to\infty} -\pi/2$

$$f(x) = \begin{cases} -\frac{\pi x}{2}, & x \le 0 \\ +\frac{\pi x}{2}, & x > 0 \end{cases}$$

* $x \neq 0$ elementarno neprekidna

$x=0$ $L_1 = L_2 = f(0) = 0$ – f-ja je neprekidna $\forall x \in R$.

(Z) Ispitati neprekidnost f-je date sa:

$$f(x) = \begin{cases} \dfrac{x^\alpha}{1+e^{-1/x}}, & x \neq 0 \\ 0, & x=0, \ \alpha \in R \end{cases}$$

1° $\alpha = 0$ prekid I vrste

2° $\alpha \neq 0 \Rightarrow$ izbor α za koje zadatak ima smisla:

$\alpha \in Z \setminus \{0\}$ → $Z_+ \in N$, → Z_-

$\alpha = \frac{p}{q}$, mjera $\frac{p}{q} = 1$ tj. relativno prosti

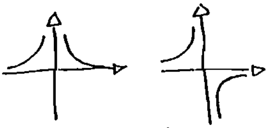
$q = 2m+1$ (neparan)

$p \in Z/\{0\}$

Ako je $\alpha \in N$ – čista neprekidnost

Ako je $\alpha = -n$ $\quad f(x) = \dfrac{1}{x^n(1+e^{-1/x})}$, $n = 1,2,\ldots$ i to vrijedi u okolini $O(0)$.

$\frac{1}{x^n}$ BMV.

$\lim_{x \to 0} |f(x)| = +\infty$ – prekid II vrste $\Rightarrow$ 

U slučaju da je $\alpha = \frac{p}{q}$, $p \in Z$ – isti kao predhodni slučaj. Za $p = Z_+$ – f-ja neprekidna u $x=0$.

predavanja

Nizovi

(Z) $\lim\limits_{(n)} \frac{1^p+2^p+3^p+\ldots+n^p}{n^{p+1}}$, $(p>-1)$

RJ: Prema Štolzovom teoremu koji glasi:

Neka su dati nizovi X_n i Y_n i neka su zadovoljeni sljedeći uslovi:

1. $Y_n \to +\infty$
2. $Y_{n+1} > Y_n$ počevši od nekog n pa nadalje
3. postoji konačan ili ∞ $\lim\limits_{(n)} \frac{X_{n+1}-X_n}{Y_{n+1}-Y_n} = L$

Tada postoji $\lim\limits_{(n)} \frac{X_n}{Y_n}$ i vrijedi $\lim\limits_{(n)} \frac{X_n}{Y_n} = L$.

1° $Y_n = n^{p+1}$, $\lim\limits_{(n)} n^{p+1} = +\infty$

2° $n^{p+1} > (n-1)^{p+1} \;/()^{\frac{1}{p+1}} \Rightarrow n > n+1$

3° $\lim\limits_{(n)} \frac{X_{n+1}-X_n}{Y_{n+1}-Y_n} = \lim\limits_{(n)} \frac{X_n - X_{n-1}}{Y_n - Y_{n-1}} = \lim\limits_{(n)} \frac{n^p}{n^{p+1}-(n-1)^{p+1}} =$

$= \lim\limits_{(n)} \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}$ – uzimamo zadnji izraz jer je njega lakše pokratiti. Izraz prije njega ima $\ominus$ $n(n-1)$ pa bi dobili niz sa $\oplus$ i $\ominus$.

$\lim\limits_{(n)} \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} = \left| \begin{array}{l} \text{Treba ga razviti po Binomnoj formuli:} \\ (n+m)^p = \sum n^{p-k} \cdot m^k \end{array} \right| =$

$\Rightarrow (n+1)^p = \binom{p}{0} n^p \cdot 1 + \binom{p}{1} n^{p-1} 1^2 + \ldots + \binom{p}{p} n^1 \cdot 1^p$

$(n+1)^{p+1} = \binom{p+1}{0} n^{p+1} \cdot 1^1 + \binom{p+1}{1} n^p \cdot 1^2 + \ldots + \binom{p+1}{p+1} n^1 \cdot 1^{p+1}$

$(n+1)^p = n^p + p \cdot n^{p-1} + \ldots + n$

$(n+1)^{p+1} = n^{p+1} + n^p(p+1) + \ldots + n$

$$\frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}} = \frac{n^p + p\cdot n^{p-1}+\ldots+n}{n^{p+1}+n^p(p+1)+\ldots+n-n^{p+1}} = \frac{n^p+p\cdot n^{p-1}+\ldots+n}{n^p(p+1)+\ldots+n} \Big/ : \frac{n^p}{n^p}$$

$$= \frac{1+p\frac{1}{n}+\ldots+\frac{1}{n^{p-1}}}{(p+1)+\ldots+\frac{1}{n^{p-1}}} \Big/ \odot \lim\limits_{(n)} = \lim\limits_{(n)} \frac{1+p\cdot\frac{1}{n}+\ldots+\frac{1}{n^{p-1}}}{(p+1)+\ldots+\frac{1}{n^{p-1}}} = \frac{1}{p+1}.$$

(Z) Ispitati konvergenciju niza: $a_n = \frac{1^2+2^2+3^2+\dots+n^2}{n^n}$

* Ovo je pogrešno zadan zadatak, ali mi ćemo ga ipak pokušati riješiti preko Štolza!

1° $y_n = n^n$, $\lim_{(n)} n^n = +\infty$ (T)

2° $(n+1)^{(n+1)} > n^n$ (T)

3° $\lim \left(\frac{n^2}{n^n - (n-1)^{n-1}} \right)$ ili $\lim_{(n)} \left(\frac{(n+1)^2}{(n+1)^{n+1} - n^n} \right) =$

$$= \lim_{(n)} \left(\frac{(n+1)^2}{n^{n+1} + n^n \frac{(n+1)}{1!} + n^{n-1} \cdot \frac{(n+1)n}{2!} + \dots + 1 - n^n} \right) =$$

$$= \lim_{(n)} \left(\frac{n^2 + 2n + 1}{n^{n+1} + n^n \left(\frac{n+1}{1!}\right) + \frac{(n+1)n}{2!} \cdot n^{n-1} + \dots + 1 - n^n} \right) \Big/ \cdot \frac{n^{n+1}}{n^{n+1}} =$$

$$= \lim_{(n)} \left(\frac{\frac{1}{n^{n-1}} + \frac{2}{n^n} + \frac{1}{n^{n+1}}}{1 + \frac{1}{n}(n+1) + \frac{1}{n}\frac{(n+1)}{2!} + \dots + \frac{1}{n^{n+1}} - \frac{1}{n}} \right) =$$

= ovako ne može.

U nazivniku imamo izraz n^n, a $n \to +\infty$ pa na ovakav izraz ne možemo primjeniti formulu za binomni obrazac. Eksponent po kojem se razvije binomna formula mora biti konstanta.

Ovaj zadatak bi trebalo riješiti pomoću neke nejednakosti.

$$a_n = \frac{1^2+2^2+\dots+n^2}{n^n} = \frac{\frac{n}{6}(2n+1)(n+1)}{n^n} = \frac{n(2n+1)(n+1)}{6n^n} = \frac{2n^3+3n^2+n}{6n^n}.$$

kada bi ovo podjelili sa n^n opet bi dobili 0.

$$\frac{1^n}{n^n} < \frac{1^2+2^2+\dots+n^2}{n^n} \le \frac{n^1+n^2+\dots+n^n}{n^n} = \frac{n(1+n+n^2+\dots+n^{n-1})}{n^n} <$$

$$< \frac{1 \cdot \frac{1-n^n}{1-n}}{n^n} \cdot n = \frac{n^n(n-1)}{n^n(n-1)} = 1 \quad \text{ili}$$

$$\frac{1^n}{n^n} < a_n \le \frac{n^2+n^2+\dots+n^2}{n^n} = \frac{n^3}{n^n} \Rightarrow$$

$$\Rightarrow \frac{1^n}{n^n} < a_n \le \frac{1}{n^{n-1}} \Rightarrow 0 < a_n < 0 \Rightarrow a_n \to 0$$

$$a_1 = \frac{1^2}{1^1} = 1, \quad a_2 = \frac{1^2+2^2}{2^2} = \frac{5}{4}, \quad a_3 = \frac{1^2+2^2+3^2}{3^3} = \frac{14}{27}$$

$$a_4 = \frac{1^2+2^2+3^2+4^2}{4^4} = \frac{30}{256}, \quad a_5 = \frac{1^2+2^2+3^2+4^2+5^2}{5^5} = \frac{55}{3125}$$

Niz je: $1, \frac{5}{4}, \frac{14}{27}, \frac{30}{256}, \frac{55}{3125}, \ldots$

Očigledno da niz teži nuli.

(Z) $a_n = \frac{1+2+3+\ldots+n}{n^2}$

$$1+2+\ldots+n = \frac{n}{2}(n+1)$$

$$a_n = \frac{n(n+1)}{2n^2} = \frac{1}{2}\left(1+\frac{1}{n}\right) \Rightarrow \lim_{(n)} \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{2}$$

Preko Štolza:

1° $n^2 \to +\infty$ kod $n \to +\infty$ (T)

2° $(n+1)^2 > n^2 \,/\sqrt{\ } \Rightarrow n+1 > n$ (T)

$$3° \lim_{(n)} \frac{x_n - x_{n-1}}{y_n - y_{n-1}} = \lim_{(n)} \frac{n}{n^2-(n-1)^2} = \lim_{(n)} \frac{n}{n^2-(n^2-2n+1)} =$$

$$= \lim_{(n)} \frac{n}{2n-1} \Big/ : \frac{n}{n} = \lim_{(n)} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

(Z) Dokazati da je niz: $\left\{ \underbrace{\left(1+\frac{1}{n}\right)^{n+1}}_{a_n} \right\}_{n=1}^{\infty}$ opadajući!

$$\frac{a_n}{a_{n-1}} = \frac{\left(1+\frac{1}{n}\right)^{n+1}}{\left(1+\frac{1}{n-1}\right)^n} = \frac{\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n}{\left(1+\frac{1}{n-1}\right)^n} = \left(\frac{\frac{n+1}{n}}{\frac{n}{n-1}}\right)^n \left(1+\frac{1}{n}\right) =$$

$$= \left(\frac{(n-1)(n+1)}{n^2}\right)^n \left(1+\frac{1}{n}\right) = \left(\frac{n^2+n-n+1^2}{n^2}\right)^n \left(\frac{n+1}{n}\right) = \underbrace{\left(\frac{n^2-1^2}{n^2}\right)}_{<1}\underbrace{\left(\frac{n+1}{n}\right)}_{>1} < 1$$

$$\frac{a_n}{a_{n-1}} < 1 \Rightarrow a_n < a_{n-1}$$

Ovo se bolje vidi ako sve izmnožimo što smo primijenimo Bernulija:

$$\left(\frac{n^2-1}{n^2}\right)^n \cdot \frac{n+1}{n} = \left(\frac{1}{1+\frac{1}{n^2-1}}\right)^n \cdot \frac{n+1}{n} = \left(\frac{1}{1+n\frac{1}{n^2-1}}\right) \cdot \frac{n+1}{n} = \text{______}$$

$$= \frac{n^3+n^2-n-1}{n^3+n^2-n} < 1.$$

(2) Dokazati da niz: $x_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n,\ n\in N$ konvergira!

Imamo: $x_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n \Rightarrow$

$\Rightarrow 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} = \ln n + c + \theta_n$

- $\theta_n \to 0,\ n\to+\infty$
- c – Ojlerova konstanta $c = 0{,}577\ldots$

Da bi dokazali da ovaj niz konvergira trebamo dokazati da je monoton i ograničen.

$$x_{n+1}-x_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}+\frac{1}{n+1}-\ln(n+1)-\left(1+\frac{1}{2}+\dots+\frac{1}{n}-\ln n\right) =$$

$$= \frac{1}{n+1}-\ln(n+1)+\ln n$$

$$\frac{1}{n+1}-\ln(n+1)+\ln n = \ln\left(\frac{n}{n+1}\right)+\frac{1}{n+1} = -\ln\left(\frac{n+1}{n}\right)+\frac{1}{n+1}$$

Uzimamo ovaj izraz.

$$\frac{1}{n+1}-\ln(n+1)+\ln(n) = \frac{1}{n+1}-\ln\left(\frac{n+1}{n}\right) = \frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)$$

kako je $\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}$ to je

$\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right) < 0 \Rightarrow$ da je: $a_{n+1} < a_n$ pa je niz monotono opadajući.

Dokažimo da je ograničen!

$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n > \ln(1+\frac{1}{1})+\ln(1+\frac{1}{2})+\dots+\ln(1+\frac{1}{n})-$

$-\ln n = \ln(2\cdot\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n})-\ln n = \ln(\frac{n+1}{n}) = \ln(1+\frac{1}{n})$

Dakle, $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n > \ln(1+\frac{1}{n}) > \underbrace{\frac{1}{n+1}} > 0$

ovo je oblast od koje je niz veći.

$\frac{1}{n+1}$ - od koje je niz $1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n$ veći $\forall n\in N$ predstavlja konačni presjek c - a on je Ojlerova konst.

Prema tome možemo pisati da je:

$1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n = c+\theta_n$, gdje je $\theta_n\to 0$, $n\to\infty$, a c je Ojlerova konstanta.

Pošto smo dokazali da je niz ograničen i monoton možemo tvrditi da je on i konvergentan!

(2) Potrebno je naći sumu S_n:

$\underbrace{S_n = \frac{1}{2}+\frac{3}{2^2}+\dots+\frac{2n-1}{2^n}}$. $\lim_{(n)}\left(\frac{1}{2}+\frac{3}{2^2}+\dots+\frac{2n-1}{2^n}\right)$!

$\frac{1}{2}S_n = \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{3}{2^2}+\dots+\frac{1}{2}\cdot\frac{2n-1}{2^n}$

$S_n-\frac{1}{2}S_n = \frac{1}{2}+\frac{3}{2^2}+\dots+\frac{2n-1}{2^n}-\frac{1}{2^2}-\frac{3}{2^3}-\dots-\frac{2n-1}{2^{n+1}} =$

$= \frac{1}{2}+\left(\frac{3}{2^2}-\frac{1}{2^2}\right)+\left(\frac{5}{2^3}-\frac{3}{2^3}\right)+\dots+\left(\frac{2n-1}{2^n}-\frac{2n-3}{2^n}\right)-\frac{2n-1}{2^{n+1}} =$

$= \frac{1}{2}+\left(\frac{3}{2^2}-\frac{1}{2^2}\right)+\dots-\frac{2n-1}{2^{n+1}} = \frac{1}{2}+\frac{2}{2^2}+\frac{2}{2^3}+\dots+\frac{2}{2^n}-\frac{2n-1}{2^{n+1}} =$

$= \frac{1}{2}+\underbrace{\left(\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}}\right)}_{\text{geometrijski red}}-\frac{2n-1}{2^{n+1}} =$

$$S_n = a_1\cdot\frac{q^n-1}{q-1} = a_1\cdot\frac{1-q^n}{1-q} = \frac{1}{2}\,\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 1-\frac{1}{2^n}$$

$= \frac{1}{2}+\sum_{k=1}^{n-1}\frac{1}{2^k}-\frac{2n-1}{2^{n+1}} = \frac{1}{2}+\sum_{k=1}^{n-1}\frac{1}{2^k}-\frac{2n-1}{2^{n+1}} =$

$$= \frac{1}{2} + \left(1 - \frac{1}{2^n}\right) - \frac{2n-1}{2^{n+1}} = \frac{1}{2} S_n \;/\cdot 2 \Rightarrow$$

$$\frac{1}{2} S_n = \frac{1}{2} + 1 - \frac{1}{2^n} - \frac{2n-1}{2^{n+1}} \;/\cdot 2 \Rightarrow$$

$$S_n = 1 + 2 - \frac{2}{2^n} - 2\cdot\frac{2n-1}{2^{n+1}} = 1 + 2 - \frac{1}{2^{n-1}} - \frac{2n-1}{2^n} =$$

$$= 1 + 2 - \frac{2-(2n-1)}{2^n} = 3 - \left(\frac{-2n+3}{2^n}\right) \;/ \lim_{(n)} \Rightarrow$$

$$\Rightarrow 3 - \lim_{(n)} \frac{n}{2^{n-1}} \overset{0}{} = \underline{\underline{3}} \,//$$

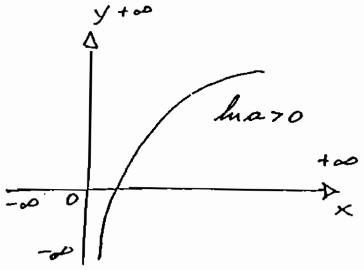
$$S_n = a_1 + a_1\cdot q + a_1\cdot q^2 + \dots + a_1\cdot q^{n-1} = a_1 \frac{1-q^n}{1-q}$$

$$\lim_{(n)} \frac{n}{2^{n-1}} = L \;/ \ln \Rightarrow \lim_{(n)} \ln \frac{n}{2^{n-1}} = \lim_{(n)} \ln n - \ln 2^{n-1} = \lim_{(n)} \left(\ln n - (n-1)\ln 2\right) = \ln L$$

$\downarrow$ konstanta.

$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n}$ — OVO TRENUTNO NE POMAŽE

$$\lim_{(n)} \ln n - \underbrace{\lim_{(n)} (n-1)\ln 2}_{\to -\infty} = \ln L$$



$$\lim_{(n)} \ln n (-\infty) = \ln L$$

$$-\infty = \ln L \;/\cdot \text{ant log}$$

$$\ln L = -\infty$$

$$\frac{1}{e^{\infty}} = L \Rightarrow L = 0_+$$

(2)

$$\lim_{(n)} \left(\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \dots + \frac{1}{n(n+1)}\right)$$

Primijetimo da $\left(\frac{1}{1\cdot 2}\right)$ možemo pisati u obliku $\left(1 - \frac{1}{2}\right)$,

$(1/2\cdot 3) - \dots - \left(\frac{1}{2} - \frac{1}{3}\right)$

$\left(\frac{1}{2\cdot 3}\right)$ možemo pisati u obliku $\left(\frac{1}{2}-\frac{1}{3}\right)$

$\left(\frac{1}{3\cdot 4}\right)$ -//- $\left(\frac{1}{3}-\frac{1}{4}\right)$

$\vdots$

$\frac{1}{n(n+1)} = \left(\frac{1}{n}-\frac{1}{n+1}\right)$

Pišemo: $\lim\limits_{(n)}\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}\right) =$

$= \lim\limits_{(n)}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right] = \lim\limits_{(n)}\left[1-\frac{1}{n+1}\right] =$

$= \lim\limits_{(n)} 1 - \lim\limits_{(n)}\frac{1}{n+1}^{\,0} = 1$ (T)

(Z) Dokazati da je: $\lim\limits_{(n)}\frac{n}{\sqrt[n]{n!}} = e$! Kako imamo korijen, naštimočemo da cio izraz bude pod korijenom, i tada ćemo upotrijebiti poznatu jednakost:

$$\lim_{(n)}\sqrt[n]{a_n} = \lim_{(n)}\frac{a_{n+1}}{a_n}.$$

Dakle, $\lim\limits_{(n)}\sqrt[n]{\frac{n^n}{n!}} = \lim\limits_{(n)}\sqrt[n]{a_n}$, $a_n = \frac{n^n}{n!} \Rightarrow$

$$\lim_{(n)}\frac{a_{n+1}}{a_n} = \lim_{(n)}\frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{(n)}\frac{(n+1)^{n+1}\cdot n!}{(n+1)!\cdot n^n} =$$

$$= \lim_{(n)}\frac{(n+1)^n\cdot(n+1)\cdot n!}{(n+1)!\cdot n^n} = \lim_{(n)}\frac{(n+1)^n(n+1)!}{(n+1)!\cdot n^n} = \lim_{(n)}\left(\frac{n+1}{n}\right)^n =$$

$$= \lim_{(n)}\left(1+\frac{1}{n}\right)^n = e \quad \text{ŠTD}$$

(Z) $\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\dots+\frac{1}{\sqrt{n^2+n}} = a_n$. Naći limes!

Najmanji član niza je: $\frac{1}{\sqrt{n^2+n}}$

Najveći član niza je: $\frac{1}{\sqrt{n^2+1}}$

U tom slučaju imamo da je:

$$\underbrace{\frac{n}{\sqrt{n^2+n}}}_{b_n} \le a_n \le \underbrace{\frac{n}{\sqrt{n^2+1}}}_{c_n}$$ → Sendvič teorem.

$$\lim_{(n)} b_n = \lim_{(n)} \frac{n}{\sqrt{n^2+n}} = \lim_{(n)} \sqrt{\frac{n^2}{n^2+n}} = \lim_{(n)} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1$$

$$\lim_{(n)} c_n = \lim_{(n)} \frac{n}{\sqrt{n^2+1}} \Big/ : \frac{n}{n} \Rightarrow \lim_{(n)} \frac{1}{\sqrt{1+\frac{1}{n^2}}} = 1 \Rightarrow$$

$$\Rightarrow \quad 1 \le a_n \le 1 \quad \Rightarrow \quad \text{dakle } a_n \to 1$$

(Z) Dokazati da je niz $x_n = 1+\frac{1}{2}+\ldots+\frac{1}{n}$, $n \in \mathbb{N}$ divergentan!

Upotrijebit ćemo Košijev teorem:

Neka je ε proizvoljan broj iz $(0,\frac{1}{2})$. Tada imamo:

$$|x_{n+p} - x_n| = \Big|1+\frac{1}{2}+\ldots+\frac{1}{n}+\underbrace{\frac{1}{n+1}+\ldots+\frac{1}{n+p}}_{p\text{-elemenata}}-x_n\Big| = \Big|\frac{1}{n+1}+\ldots+\frac{1}{n+p}\Big| =$$

$$= \underbrace{\frac{1}{n+1}}_{\text{najveći}}+\frac{1}{n+2}+\ldots+\underbrace{\frac{1}{n+p}}_{\text{najmanji}} > p \cdot \frac{1}{n+p}$$

Za $n=p \Rightarrow n \cdot \frac{1}{n+n} = \frac{1}{2}$ tj., imamo:

$|x_{n+p} - x_n| \geq \frac{1}{2} > \varepsilon$ pa niz divergira u širem smislu.

(Z) $x_n = \frac{1}{\ln 2} + \frac{1}{\ln 3} + \ldots + \frac{1}{\ln n}$, $n \in \mathbb{N}$

Neka je ε proizvoljan iz intervala $[0,\frac{1}{2}]$. Primjenimo Košija: $|x_{n+p} - x_n| = \Big|\frac{1}{\ln(n+1)} + \frac{1}{\ln(n+2)} + \ldots + \frac{1}{\ln(n+p)}\Big| = \Big|\begin{matrix}\text{iz:}\\ n+p > \ln(n+p)\end{matrix}\Big| =$

$$= \frac{1}{\ln(n+1)} + \frac{1}{\ln(n+2)} + \ldots + \frac{1}{\ln(n+p)} > p \cdot \frac{1}{\ln(n+p)} = \frac{p}{\ln(n+p)} \geq \frac{p}{n+p}$$

Za $p=n$ imamo $\frac{n}{n+p} = \frac{n}{2n} = \frac{1}{2} > \varepsilon$ pa niz divergira.

(Z) Dokazati da je $\lim_{(n)} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{kn} \right) = \ln k$!

kako je $\frac{1}{n+1} < \left(\ln\left(1+\frac{1}{n}\right)\right) < \frac{1}{n}$ to je:

$$a_n < \ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{1}{n+1}\right) + \dots + \ln\left(1+\frac{1}{kn-1}\right) =$$

$$= \ln\left(\frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdot \dots \cdot \frac{kn}{kn-1} \right) = \ln \frac{kn}{n} = \ln k$$

$$a_n > \ln\left(\frac{n+2}{n+1} \cdot \dots \cdot \frac{kn}{kn-1} \cdot \frac{kn+1}{kn} \right) = \ln \frac{kn+1}{n+1} = \ln k, \quad n \to +\infty$$

Zbog ovoga je: $\ln k < a_n < \ln k$ pa $a_n \to \ln k$, $n \to +\infty$.

(Z) $$\lim_{(n)} \left[\frac{1}{n^3} + \frac{2^2}{n^3} + \dots + \frac{(n-1)^2}{n^3} \right] = \frac{1}{n^3} \lim_{(n)} \left[1 + 2^2 + 3^2 + \dots + (n-1)^2 \right] =$$

$$= \frac{1}{n^3} \lim_{(n)} \left[\frac{n}{6}(2n+1)(n+1) - n^2 \right] = \frac{1}{n^3} \lim_{(n)} \left[\frac{2n^3 - 3n^2 + n}{6} \right] =$$

$$= \lim_{(n)} \left(\frac{2n^3}{6n^3} - \frac{3n^2}{6n^3} + \frac{n}{6n^3} \right) = \frac{1}{3}$$

(Z) $$\lim_{(n)} \left[\frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \dots + (-1)^{n-1} \frac{n}{n} \right]$$

$$\lim_{(n)} \left[\frac{1}{n} - \frac{2}{n} + \frac{3}{n} - \dots + (-1)^{n-1} \frac{n}{n} \right] = \lim_{(n)} \frac{1}{n} \left[1 - 2 + 3 - \dots + (-1)^{n-1} n \right] \Rightarrow$$

$\Rightarrow$

$\underline{\underline{n = 2k}}$

$$\lim_{(n)} \frac{1}{2k} \left[1 - 2 + 3 - \dots + 2k-1 - 2k \right] = \lim_{(n)} \frac{1}{2k} \left[\underbrace{(1+3+\dots+2k-1)}_{\frac{k}{2}(1+2k-1) = k^2} - \underbrace{(2+4+\dots+2k)}_{2(1+2+3+\dots+k) = 2\frac{k}{2}(k+1) = k(k+1)} \right]$$

$$\lim_{(n)} \frac{1}{2k} \left[k^2 - k^2 - k \right] = \lim_{(n)} \left(-\frac{1}{2} \right) = -\frac{1}{2}.$$

$\underline{\underline{n = 2k-1}}$

1° $\lim\limits_{(n)} \frac{1}{2k-1}\left[1-2+3-4+5-\ldots-2k-2+2k-1\right] =$

$= \lim\limits_{(n)} \frac{1}{2k-1}\left[\underbrace{(1+3+5+\ldots+2k-1)}_{\frac{k}{2}(1+2k-1)=k^2} - \underbrace{(2+4+6+\ldots+(2k-2))}_{2(1+2+3+\ldots+(k-1)) = 2\left(\frac{k}{2}(1+k-1)\right) = k^2}\right] =$

$= \lim\limits_{(n)} \frac{1}{2k-1}\cdot 0 = 0$

2° $\lim\limits_{(n)} \frac{1}{2k-1}\left[\underbrace{(1+3+\ldots+(2k+1))}_{\frac{k}{2}(1+2k+1)=k^2+k} - \underbrace{(2+4+\ldots+2k+2)}_{2(1+2+3+\ldots+k+1) = 2\cdot\frac{k}{2}(k+1+1) = k(k+2) = k^2+2k}\right] =$

$\lim\limits_{(n)} \frac{1}{2k-1}\left[k^2+k-k^2-2k\right] = \lim\limits_{(n)} \frac{-k}{-(1-2k)} \Big/ : \frac{k}{k} =$

$= \lim\limits_{(n)} \frac{-1}{-\left(\frac{1}{n}-2\right)} = \underline{\frac{1}{2}}.$

Azemov način sa vježbi:

za $n=2k$ imamo:

$1+2+3+\ldots+k+k+1+\ldots+2k-2(2+4+\ldots+2k) =$

$= \underbrace{1+2+3+\ldots+k}_{+\frac{k}{2}(k+1)} + \underbrace{k+1+\ldots+2k}_{+\frac{k}{2}(k+1+2k)} - \underbrace{4(1+2+3+\ldots+k)}_{-4\cdot\frac{k}{2}(k+1)} =$

$= \frac{k}{2}\left(k+1+k+1+2k-4(k+1)\right) = \frac{k}{2}(4k+2-4k-4) = \frac{k}{2}(-2) = \underline{-k}.$

$\lim\limits_{(n)} \frac{1}{n}\left[1-2+3-\ldots+(-1)^{n-1}\cdot n\right] = \left| \text{za } n=2k-1 \right| =$

$= \lim\limits_{(n)} \frac{1}{2k-1}\left[1-2+3-\ldots+(-1)^{2k-1-1}\cdot(2k-1)\right] = \frac{1}{2k-1}\left[1-2+3-\ldots+(-1)^{(2k-2)}(2k-1)\right] =$

$= \lim\limits_{(n)} \frac{1}{2k-1}\left[1-2+3-\ldots+(-1)^{(2k-2)}\cdot(2k-1)\right]$

za $n=1 \Rightarrow \frac{1}{2-1}\left[1-2+3-\ldots+(-1)^0(2k-1)\right]$

DA BI OVDJE DOBILI 2 TREBA DA STAVIMO 2k+1, jer 2k pravi 1.

Tako ćemo dobiti ovo:

$$\frac{1}{2k+1}\left[1-2+3-\ldots+(-1)^{2k}2k+1\right] \quad \text{za } k=1 \Rightarrow (-1)^2$$

$$B_{2n+1} = B_{2n} + 2n+1 = -k+2k+1 = \underline{k+1}.$$

(2) Dokazati da je:

$$\lim_{(n)}\left(\frac{1}{0!}+\frac{1}{1!}+\ldots+\frac{1}{n!}\right) = \sum_{n=0}^{\infty}\frac{1}{n!} = e$$

Treba imati na umu da je $\lim_{(n)}\left(1+\frac{1}{n}\right)^n = e$.

$$\left(1+\frac{1}{n}\right)^n = \underbrace{\underbrace{1+1+\frac{n(n-1)}{2!}\cdot\frac{1}{n^2}+\ldots+\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}\cdot\frac{1}{n^k}}_{II}+\ldots+\frac{n(n-1)\cdot\ldots\cdot 1}{n!}\cdot\frac{1}{n^n}}_{I = x_n}$$

Očigledno je da je $I > II$!

$$\left(1+\frac{1}{n}\right)^n > 1+1+\frac{n(n-1)}{2!}\frac{1}{n^2}+\ldots+\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{k!}\cdot\frac{1}{n^k} \quad \Big/ \circ \lim_{(n)}$$

$$e \geq 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{k!} \quad \text{za } k=n \Rightarrow$$

$$e > 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!} = y_n$$

↳ trebamo dokazati da $y_n \to e$ kad $n \to \infty$.

1° $e > y_n$

2° $y_n > x_n = \left(1+\frac{1}{n}\right)^n$

zbog $1+1+\frac{1}{2!}+\ldots+\frac{1}{n!} > 1+1+\underbrace{\frac{n(n-1)}{n^2}}_{<1}\cdot\frac{1}{2!}+\underbrace{\frac{n(n-1)\cdot\ldots\cdot(n-k+1)}{n^k}}_{<1}\cdot\frac{1}{k!}+\underbrace{\frac{n(n-1)\cdot\ldots\cdot 1}{n^n}}_{<1}\cdot\frac{1}{n!}$

$$e > y_n > x_n = \left(1+\frac{1}{n}\right)^n \Big/ \cdot \lim_{(n)} \quad \Rightarrow \quad y_n \to e,\ n \to +\infty.$$

(Z) $a_n = 1 + \frac{2}{3} + \frac{3}{3^2} + \dots + \frac{n+1}{3^n} \quad / :3$

$$\frac{a_n}{3} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots + \frac{n+1}{3^{n+1}}$$

$$a_n - \frac{a_n}{3} = 1 + \frac{2}{3} + \frac{3}{3^2} + \dots + \frac{n+1}{3^n} - \left(\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \dots + \frac{n+1}{3^{n+1}}\right) =$$

$$= 1 - \frac{1}{3} + \frac{2}{3} - \frac{2}{3^2} + \frac{3}{3^2} - \frac{3}{3^3} + \frac{4}{3^3} - \dots + \frac{n+1}{3^n} - \frac{n+1}{3^{n+1}} =$$

$$= 1 - \frac{n+1}{3^{n+1}} + \underbrace{\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \dots + \frac{1}{3^n}}_{\text{geometrijski red}} = 1 - \frac{n+1}{3^{n+1}} + \left(\frac{1}{2} - \frac{1}{2\cdot 3^n}\right) \Rightarrow$$

geometrijski red

$$S_n = \frac{1}{3} \cdot \frac{1 - \frac{1}{3^n}}{\frac{2}{3}} = \frac{1}{3}\underbrace{\left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}\right)}_{S_n}$$

$$\frac{2a_n}{3} = 1 - \frac{n+1}{3^{n+1}} + \frac{1}{2} - \frac{1}{2\cdot 3^n}$$

$$a_n = \frac{3}{2} - \frac{3}{2}\frac{n+1}{3^{n+1}} + \frac{3}{4} - \frac{3}{4\cdot 3^n} \quad / \lim_{(n)}$$

$$\lim_{(n)}\left(\frac{3}{2} + \frac{3}{4} - \left(\frac{3}{2}\frac{n+1}{3^{n+1}} + \frac{3}{4\cdot 3^n}\right)\right) = \frac{3}{2} + \frac{3}{4} = \frac{9}{4}$$

(Z) $a_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$. Dokazati da konvergira!

Treba pokazati da je niz ograničen i monoton.

$$a_{n+1} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1}$$

$$a_n = 1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} - 2\sqrt{n}$$

$$a_{n+1} - a_n = \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n} = \frac{1}{\sqrt{n+1}} - 2\sqrt{n+1} + 2\sqrt{n} =$$

$$= \frac{1}{\sqrt{n+1}} - 2\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{1}{\sqrt{n+1}} - \left(\frac{2(n+1-n)}{\sqrt{n+1} + \sqrt{n}}\right) = \frac{\sqrt{n} - \sqrt{n+1}}{\sqrt{n+1}\left(\sqrt{n+1} + \sqrt{n}\right)} < 0$$

Niz je monotono opadajući!

Tražimo ograničenje s donje strane:

$\sqrt{n+1}-\sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{2\sqrt{n}} \Rightarrow \sqrt{n+1}-\sqrt{n} < \frac{1}{2\sqrt{n}}$

Za $n=1 \Rightarrow \sqrt{2}-\sqrt{1} < \frac{1}{2}$ $/\cdot 2$

$n=2$ $\quad \sqrt{3}-\sqrt{2} < \frac{1}{2\sqrt{2}}$ $/\cdot 2$

$n=3$ $\quad \sqrt{4}-\sqrt{3} < \frac{1}{2\sqrt{3}}$ $/\cdot 2$

$\vdots$

$n=n$ $\quad \sqrt{n+1}-\sqrt{n} < \frac{1}{2\sqrt{n}}$ $/\cdot 2$

$2\sqrt{2}-2 < 1$

$2\sqrt{3}-2\sqrt{2} < \frac{1}{\sqrt{2}}$

$2\sqrt{4}-2\sqrt{3} < \frac{1}{\sqrt{3}}$

$\vdots$

$2\sqrt{n+1}-2\sqrt{n} < \frac{1}{\sqrt{n}}$ $\quad \oplus$

$2\sqrt{2}-2+2\sqrt{3}-2\sqrt{2}+2\sqrt{4}-2\sqrt{3}+\dots+2\sqrt{n+1}-2\sqrt{n} < 1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}$

$-2+2\sqrt{n+1} < 1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}$ $\quad /-2\sqrt{n}$

$-2-2\sqrt{n}+2\sqrt{n+1} < \underbrace{1+\frac{1}{\sqrt{2}}+\dots+\frac{1}{\sqrt{n}}-2\sqrt{n}}$

$-2-2\sqrt{n}+2\sqrt{n+1} < a_n \Rightarrow a_n > -2+2\underbrace{(\sqrt{n+1}-\sqrt{n})}_{>0} > -2$

Kako je monotono opadajući i ograničen s donje strane niz konvergira!

② Dokazati da je $\lim_{(n)} \sqrt[n]{a} = 1$!

za $a>1 \Rightarrow a^{\frac{1}{n}}-1 = h_n$, $h_n>0$, $n=1,2,3,\dots$

↳ jer je $a^{\frac{1}{n}}>1$ pa je $a^{\frac{1}{n}}-1>0$

Treba pokazati da za dati $\varepsilon>0$ $\exists$ $n_0(\varepsilon)\in\mathbb{N}$ takav da vrijedi: $\quad |a^{\frac{1}{n}}-1| = |h_n| < \varepsilon$.

$|a^{\frac{1}{n}}-1| = |h_n| \Rightarrow a^{\frac{1}{n}} = h_n+1$

$a^{\frac{1}{n}} = (h_n+1)$ $/\odot$ stepenujemo

$a = (h_n+1)^n = 1+nh_n+\binom{n}{2}h_n^2+\dots+h_n^n > nh_n$

$a > nh_n = h_n < \frac{a}{n} = N(\varepsilon)$ pa vrijedi:

$|a^{\frac{1}{n}}-1| < \varepsilon$ $\Delta \equiv\!\!> \quad \underline{h_n < \frac{a}{n} = N(\varepsilon) < \varepsilon \quad \forall n \geq N(\varepsilon)}$

Može i na drugi način:

Za $a>1 \Rightarrow \sqrt[n]{a}>1$

Za $n>a \Rightarrow \sqrt[n]{n} > \sqrt[n]{a}$ Dakle,

$1 < \sqrt[n]{a} < \sqrt[n]{n} \quad / \cdot \lim\limits_{(n)}$

$\lim\limits_{(n)} \sqrt[n]{n} = \lim\limits_{(n)} \frac{n+1}{n} = 1$ pa je $1 < \sqrt[n]{a} < 1 \Rightarrow$

$\Rightarrow \lim\limits_{(n)} \sqrt[n]{a} \to 1$

za $0<a<1$

$|a| = \frac{1}{|b|}$, gdje je $|b|>1$ pa je:

$\sqrt[n]{a} = \sqrt[n]{\frac{1}{b}} = \frac{1}{\sqrt[n]{b}} \Rightarrow \lim\limits_{(n)} \sqrt[n]{\frac{1}{b}} = \lim\limits_{(n)} \sqrt[n]{a} = 1$

(2) Dokazati da za sve realne x vrijedi:

$\lim\limits_{(n)} \frac{x^n}{n!} = 0$.

1° Za $|x|<1$ znamo da je $\lim\limits_{(n)} x^n = 0$

kako je $0 < \frac{1}{n!} < \frac{1}{n}$ za $\forall n \in \mathbb{N}$ što je $\frac{1}{n!}$ nula niz.

Zbog toga je $\lim\limits_{(n)} \frac{x^n}{n!} = \lim\limits_{(n)} \frac{1}{n!} \cdot \lim x^n = 0$.

2° Za $|x|=1 \Rightarrow \lim\limits_{(n)} \frac{1}{n!} = 0$ pa je tvrđenje tačno za $|x|\leq 1$

3° $|x|>1$ i k takov prirodan broj da je

$k \leq |x| < k+1$ odnosno $\frac{|x|}{k+1} < 1$

Za $n>k$ imamo: $\left|\frac{x^n}{n!}\right| = \frac{|x|^k}{k!} \cdot \frac{|x|}{k+1} \cdot \frac{|x|}{k+2} \cdots \frac{|x|}{n}$

(ispod zagrade: najveći element)

kako je $\frac{|x|}{k+l} < \frac{|x|}{k+1}$ $(l = 2,3,4,\ldots, n-k) \Rightarrow$

$\left|\left(\frac{x^n}{n!}\right)\right| < \frac{|x|^k}{k!}\left(\frac{|x|}{k+l}\right)^{n-k}$, kako je međutim $\frac{|x|}{k+l} < 1 \Rightarrow$

$\Rightarrow \left(\frac{|x|}{\ldots}\right)^{n-k}$ [illegible]

Ⓩ Ako je $\{a_n\}$, $n=0,1,2,\dots$ monotono rastući niz, naći limes:

$$\lim_{(n)}\left\{\frac{a_1}{a_0\cdot S_1}+\frac{a_2}{S_1\cdot S_2}+\frac{a_3}{S_2\cdot S_3}+\dots+\frac{a_n}{S_{n-1}S_n}\right\}$$

gdje je: $S_n=\sum_{k=0}^{n}a_k$.

RJ: Za $a_i>0 \wedge a_{n+1}>a_n \wedge S_n=\sum_0^n a_k$

kako je: $S_n=\underbrace{a_1+a_2+\dots+a_{n-1}}+a_n$ što je

$$\underbrace{S_n = S_{n-1}+a_n}$$

$$S_n=S_{n-1}+a_n \Rightarrow \boxed{S_n-S_{n-1}=a_n} \quad (1)$$

Prema formuli (1) imamo: $\left.\begin{array}{l} S_1-S_0=a_1\\ S_2-S_1=a_2\\ \vdots\\ S_n-S_{n-1}=a_n\end{array}\right\}\Rightarrow$

$$\Rightarrow \lim_{(n)}\left(\frac{S_1-S_0}{S_0\cdot S_1}+\frac{S_2-S_1}{S_1\cdot S_2}+\dots+\frac{S_n-S_{n-1}}{S_{n-1}\cdot S_n}\right)=$$

$$=\lim_{(n)}\left(\frac{1}{S_0}-\frac{1}{S_1}+\frac{1}{S_1}-\frac{1}{S_2}+\dots+\frac{1}{S_{n-1}}-\frac{1}{S_n}\right)=\lim_{(n)}\left(\frac{1}{S_0}-\frac{1}{S_n}\right)=$$

$$=\lim_{(n)}\frac{1}{a_0}-\lim_{(n)}\frac{1}{S_n}=\left|\text{za } a_i>0\Rightarrow \frac{1}{a_0+a_1+\dots+a_n}=\frac{1}{S_n}\nearrow^{0}\right|=$$

$$=\lim_{(n)}\frac{1}{a_0}-\lim_{(n)}\frac{1}{S_n}\nearrow^{0}=\underline{\frac{1}{a_0}}.$$

Ⓩ Ako postoji $\lim_{(n)} a_n$, konačan ili ∞, tada je:

$$\lim_{(n)}\frac{a_1+a_2+a_3+\dots+a_n}{n}=\lim_{(n)}a_n \qquad \text{Dokazati!}$$

Upotrijebimo Štolza!

$\left.\begin{array}{ll} 1^\circ\ y_n\to+\infty,\ n\to\infty & \checkmark\\ 2^\circ\ y_{n+1}>y_n \Rightarrow n+1>n & \checkmark\\ 3^\circ\ \lim_{(n)}\frac{a_n}{n-(n-1)}=\lim_{(n)}a_n & \end{array}\right\}$ uslovi zadovoljeni

Prema Štz. teoremu je sada: $\lim_{(n)}\frac{a_n+a_{n-1}+\dots+a_1}{n}=\lim_{(n)}a_n$.

(Z) Dokazati da niz: $a_n=\left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\cdots\left(1+\frac{1}{n^3}\right)$ konvergira!

$\ln a_n = \ln\left(1+\frac{1}{1^3}\right)+\ln\left(1+\frac{1}{2^3}\right)+\dots+\ln\left(1+\frac{1}{n^3}\right)$

kako je: $\ln\left(1+\frac{1}{n}\right)<\frac{1}{n} \Rightarrow \ln\left(1+\frac{1}{n^3}\right)<\frac{1}{n^3}$, $n\geq 1$

Tako je: $\ln a_n < \frac{1}{1^3}+\frac{1}{2^3}+\dots+\frac{1}{n^3} < \frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2} <$

$< 1+\left(\frac{2-1}{1\cdot 2}\right)+\left(\frac{3-2}{3\cdot 2}\right)+\left(\frac{4-3}{4\cdot 3}\right)+\dots+\left(\frac{n-(n-1)}{n(n-1)}\right)=$

$= 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\dots+\left(\frac{1}{n-1}-\frac{1}{n}\right) = 1+1-\frac{1}{n} = 2-\frac{1}{n} < 2$

pa je $\ln a_n < 2$.

$a_{n+1} > a_n$ pa je niz ograničen odozgo i monotono rastući pa konvergira.

(Z) $\lim\limits_{(n)}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\dots+\frac{(n-1)^2}{n^3}\right]$

$\lim\limits_{(n)}\left[\frac{1^2}{n^3}+\frac{2^2}{n^3}+\frac{3^2}{n^3}+\dots+\frac{(n-1)^2}{n^3}\right]=\lim\limits_{(n)}\frac{1}{n^3}\left[1^2+2^2+3^2+\dots+(n-1)^2\right]=$

$=\lim\limits_{(n)}\frac{1}{n^3}\left[\frac{n}{2}(2n+1)(n+1)-n^2\right]=\lim\limits_{(n)}\frac{1}{n^3}\left[\frac{2n^3-3n^2+n}{6}\right]=$

$=\lim\limits_{(n)}\frac{2n^3-3n^2+n}{6n^3}\Big/\cdot\frac{n^3}{n^3}=\lim\limits_{(n)}\frac{2-\overset{\nearrow 0}{\frac{3}{n}}+\overset{\nearrow 0}{\frac{1}{n^2}}}{6}=\underline{\frac{1}{3}}.$

(Z) $a_n=\left(1+\frac{\alpha}{2^2}\right)\left(1+\frac{\alpha}{3^3}\right)\cdots\left(1+\frac{\alpha}{n^n}\right)$, $-1\leq\alpha$

Ako je $a_n>1$ svi članovi su veći od 1.

$a_{n+1}=a_n\underbrace{\left(1+\frac{\alpha}{(n+1)^{n+1}}\right)}_{>1}>a_n$

$\ln a_n = \ln\left(1+\frac{\alpha}{2^2}\right)+\ln\left(1+\frac{\alpha}{3^3}\right)+\dots+\ln\left(1+\frac{\alpha}{n^n}\right)$

kako je: $\frac{1}{n+1}<\ln\left(1+\frac{1}{n}\right)<\frac{1}{n}$ to je:

$\ln a_n < \frac{\alpha}{2^2}+\frac{\alpha}{3^3}+\dots+\frac{\alpha}{n^n} < \frac{\alpha}{2^2}+\frac{\alpha}{2^3}+\dots+\frac{\alpha}{2^n}=$

$= \alpha\left(\frac{1}{2}\left(\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}}\right)\right) = \frac{\alpha}{2} - \frac{\alpha}{2^{n+1}}$

$\ln a_n < \frac{\alpha}{2} - \frac{\alpha}{2^{n+1}} \Rightarrow \log_e a_n < \left(\frac{\alpha}{2} - \frac{\alpha}{2^{n+1}}\right)$

$\frac{e^{\frac{\alpha}{2}}}{e^{\frac{\alpha}{2^{n+1}}}} > a_n \Rightarrow e^{\frac{\frac{\alpha}{2}}{\frac{\alpha}{2^{n+1}}}} > a_n \Rightarrow \underline{e^{2^n} > a_n}$.

(Z) Dokazati da je broj e iracionalan!

Pretpostavimo da je e racionalan broj, tj. da je $e = \frac{p}{q}$, $(p, q \in Z)$ upotrijebimo relaciju:

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} + \frac{\theta_n}{n!\,n}$, $\boxed{0 < \theta_n < 1}$

Stavimo da je $n = q$, tada ćemo imati:

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{q!} + \frac{\theta}{q! \cdot q} = \frac{p}{q}$ Dakle,

$\frac{p}{q} = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{q!} + \frac{\theta}{q! \cdot q!} \quad / \cdot q! \cdot q$

$p \cdot q! = q! \cdot q + \frac{q! \cdot q}{1!} + \frac{q! \cdot q}{2!} + \dots + q + \theta_n$

$\theta_n = \underbrace{p \cdot q! - q! q - \frac{q! \cdot q}{1!} - \frac{q! \cdot q}{2!} - \dots - q}$

Ovo je cijeli broj jer su svi sabirci cijeli brojevi, pa i θ_n mora biti cijeli broj, a pretpostavka je bila da je θ_n, $0 < \theta_n < 1$.

Pretpostavka da je e racionalan je kontradikcija pa je en iracionalan broj!

(Z) Dokazati da je za $a_n > 0$, $\lim_{(n)} \sqrt[n]{a_n} = \lim_{(n)} \frac{a_{n+1}}{a_n}$! ako $\exists \lim_{(n)} \frac{a_{n+1}}{a_n}$.

$\lim_{(n)} \sqrt[n]{a_n} = \lim_{(n)} \sqrt[n]{a_1 \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \frac{a_4}{a_3} \cdots \frac{a_n}{a_{n-1}}} = \lim_{(n)} \frac{a_n}{a_{n-1}}$.

(Z) $a_n = \dfrac{\binom{n}{0}\frac{1}{1} + \binom{n}{1}\frac{1}{2} + \dots + \binom{n}{n}\frac{1}{n+1}}{2^n}$

$a_n = \dfrac{S_n}{2^n}$, $S_n = \binom{n}{0}\frac{1}{1} + \binom{n}{1}\frac{1}{2} + \dots + \frac{1}{n+1}\binom{n}{n}$ $/\cdot(n+1)$

$S_n(n+1) = \binom{n}{0}\frac{n+1}{1} + \binom{n}{1}\frac{n+1}{2} + \dots + \frac{n+1}{n+1}\binom{n}{n} = (n+1) + \binom{n+1}{2}\cdot n + \dots + \binom{n+1}{n+1}$

kako je $\binom{n+1}{k+1} = \binom{n+1}{k+1}\binom{n}{k}$ iz $\dfrac{(n+1)!}{(k+1)!(n-k)!} = \binom{n+1}{k+1}\cdot\dfrac{n!}{k!(n-k)!}$

$S_n\cdot(n+1) = \underbrace{\binom{n+1}{1} + \binom{n+1}{2} + \dots + \binom{n+1}{n+1} + \binom{n+1}{0}}_{2^{n+1}} - 1 = 2^{n+1} - 1$

(where $\binom{n+1}{0} - 1 = 1 - 1 = 0$)

Ovo se dobije iz:

$$(1+x)^{n+1} = \binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \dots + \binom{n+1}{n+1}x^{n+1} \quad \text{za } x=1.$$

$$a_n = \frac{S_n}{2^n} = \frac{2^{n+1}-1}{2^n(n+1)} = \frac{2^{n+1}}{2^n(n+1)} - \frac{1}{2^n(n+1)} = \frac{2}{n+1} \to 0.$$

(Z) Ako postoji $\lim\limits_{(n)} a_n$ konačan ili ∞ gdje je $a_n > 0$ za $n \in \mathbb{N}$, tada je $\lim\limits_{(n)} \sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n} = \lim\limits_{(n)} a_n$.

$A_n = \sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n}$

$\ln A_n = \frac{1}{n}(\ln a_1 + \ln a_2 + \dots + \ln a_n)$ $/\lim\limits_{(n)}$

$\lim\limits_{(n)} \ln A_n = \lim\limits_{(n)}\left(\dfrac{\ln a_1 + \ln a_2 + \dots + \ln a_n}{n}\right) = |\text{ŠTOLZ}| =$

$= \lim\limits_{(n)} \dfrac{\ln a_n}{n-(n-1)} = \lim\limits_{(n)} \ln a_n \Rightarrow \lim\limits_{(n)} \ln A_n = \lim\limits_{(n)} \ln a_n$ / a nt. ln

$\Rightarrow \lim\limits_{(n)} \sqrt[n]{a_1\cdot a_2\cdot\ldots\cdot a_n} = \lim\limits_{(n)} a_n$

(Z) $\lim\limits_{(n)} \dfrac{1 + \sqrt{2} + \sqrt[3]{3} + \dots + \sqrt[n]{n}}{n}$.

$|\text{ŠTOLZ}| = \lim\limits_{(n)} \dfrac{\sqrt[n]{n}}{n-(n-1)} = \lim\limits_{(n)} \sqrt[n]{n} = \left|\begin{matrix} a_n = n \\ a_{n+1} = n+1 \end{matrix}\right| =$

$= \lim\limits_{(n)} \dfrac{a_{n+1}}{a_n} = \lim\limits_{(n)} \dfrac{n+1}{n} = \lim\limits_{(n)} \left(1 + \frac{1}{n}\right) = 1$ (where $\frac{1}{n} \to 0$)

(Z) $\lim_{(n)} \sqrt[n]{\left(1+\frac{1}{1}\right)^1\left(1+\frac{1}{2}\right)^2\left(1+\frac{1}{3}\right)^3\cdots\left(1+\frac{1}{n}\right)^n}$

$$\lim_{(n)} \sqrt[n]{2^1\cdot\left(\frac{3}{2}\right)^2\cdot\left(\frac{4}{3}\right)^3\cdot\left(\frac{5}{4}\right)^4\cdots\left(\frac{n+1}{n}\right)^n} = \lim_{(n)} \sqrt[n]{a_n} = \lim_{(n)} \frac{a_{n+1}}{a_n} =$$

$$= \left| \begin{array}{l} a_n = 2\cdot\left(\frac{3}{2}\right)^2\cdot\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n \\ a_{n+1} = 2\cdot\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+2}{n+1}\right)^{n+1} \end{array} \right| =$$

$$= \lim_{(n)} \left(\frac{n+2}{n+1}\right)^{n+1} = \lim_{(n)} \left(1+\frac{1}{n+1}\right)^{n+1} = e$$

(Z) Štolzova teorema

Neka je $b_n \to \infty$, $n \to \infty$, pri čemu je počev od nekog n, $b_{n+1} > b_n$. Tada je $\lim_{(n)} \frac{a_n}{b_n} = \lim_{(n)} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$, ako $\exists \lim_{(n)} \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$ konačan ili beskonačan.

(Z) Ispitati konvergenciju niza i naći njegov limes! $\left\{\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{2n}\right\}$

Da bi proverili konvergenciju potrebno je da ispitamo ograničenost i monotonost niza!

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}$$

$$a_{n+1} = \underbrace{\frac{1}{n+1+1} + \frac{1}{n+2+1} + \dots + \frac{1}{2n+1}}_{a_n} + \frac{1}{2n+2}$$

$$a_n + \frac{1}{2n+2}$$

$$a_{n+1} = a_n + \frac{1}{2n+2} \Rightarrow a_n \not< a_{n+1} \Rightarrow \underline{a_n > a_{n+1}}$$

opadajući.

Treba tražiti donje ograničenje!

$a_1 = \frac{1}{1+1}$

$a_2 = \frac{1}{2+1} + \frac{1}{2+2}$

$a_3 = \frac{1}{3+1} + \frac{1}{3+2} + \frac{1}{3+3}$

$\vdots$

$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n}$

$$\boxed{\frac{1}{n+1} < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}}$$

$$a_n < \ln\left(1+\frac{1}{n}\right) + \ln\left(1+\frac{1}{n+1}\right) + \dots + \ln\left(1+\frac{1}{n+n}\right) =$$

$$= \ln\left(\frac{n+1}{n} \cdot \frac{n+2}{n+1} \cdot \ldots \cdot \frac{2n}{2n-1}\right) = \ln\frac{2n}{n} = \ln 2$$

1^0 $\underline{a_n < \ln 2}$

$$a_n > \ln\frac{n+2}{n+1} + \ln\frac{n+3}{n+2} + \dots + \ln\frac{2n+1}{2n} =$$

$$= \ln\left(\frac{n+2}{n+1} \cdot \frac{n+3}{n+2} + \dots \ln\frac{2n+1}{2n}\right) = \ln\frac{2n+1}{n+1} \Big/ \cdot \lim_{(n)}$$

$$\lim a_n > \ln \lim_{(n)} \frac{2n+1}{n+1} = \ln 2 \Rightarrow$$

$$\ln 2 < \lim_{(n)} a_n < \ln 2 \Rightarrow \lim_{(n)} a_n = \underline{\ln 2}.$$

(2) Dat je niz $a_0 = -\ln 4$, $a_n = \ln\frac{(n+1)(n+3)}{n(n+4)}$, $n = 1, 2, \dots$
Ako je $S_n = n(a_0 + a_1 + \dots + a_n)$ naći $\lim_{(n)} S_n$!

$$\left.\begin{aligned} a_0 &= -\ln 4 \\ a_1 &= \ln\frac{2\cdot 4}{1\cdot 5} \\ a_2 &= \ln\frac{3\cdot 5}{2\cdot 6} \\ a_3 &= \ln\frac{4\cdot 6}{3\cdot 7} \\ a_4 &= \ln\frac{5\cdot 7}{4\cdot 8} \\ &\vdots \\ a_n &= \ln\frac{(n+1)(n+3)}{n(n+4)} \end{aligned}\right\}$$

$$S_n = n\left(\ln\left(\frac{2\cdot 4}{1\cdot 5} \cdot \frac{3\cdot 5}{2\cdot 6} \cdot \frac{4\cdot 6}{3\cdot 7} \cdots \frac{(n+1)(n+3)}{n(n+4)}\right) - \ln 4\right) =$$

$$= n\left(\ln\frac{4(n+1)}{(n+4)\cdot 4}\right) = n\ln\frac{(n+1)}{n+4} \Rightarrow$$

$$\lim_{(n)} S_n = \lim_{(n)} n\ln\left(1 - \frac{3}{n+4}\right) = \underline{-3}.$$

② Dat je niz $\{a_n\}$ gdje je:

$$a_n = \frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}} + \left(\frac{n+1}{n-1}\right)^n + (1+\alpha)\sin\frac{n\pi}{2}.$$ Odrediti α iz uslova da niz konvergira i naći gr. vrijednost!

RJ: Prema izreci slavnog matematičara čijeg imena se ne sjećam: „Ako niz konvergira, tada mu i svaki podniz konvergira", Zaključujemo da svi podnizovi moraju konvergirati konačnom broju.

Δ $$\lim_{(n)} \left(\frac{3^n + (-2)^n}{3^{n+1} + (-2)^{n+1}}\right) = \lim_{(n)} \left(\frac{3^n}{3^{n+1} + (-2)^{n+1}} + \frac{(-2)^n}{3^{n+1} + (-2)^{n+1}}\right) =$$

$$= \lim_{(n)} \left(\frac{\frac{1}{3}}{1 + \left(-\frac{2}{3}\right)^{n+1}} + \frac{-\frac{1}{2}}{1 + \left(\frac{3}{2}\right)^{n+1}}\right) =$$

$$= \left| \begin{array}{l} \left|-\frac{2}{3}\right| < 1 \text{ pa } \left(-\frac{2}{3}\right)^{n+1} \to 0,\ n \to +\infty \\ \left|\frac{3}{2}\right| > 1 \text{ pa } \left(\frac{3}{2}\right)^{n+1} \to +\infty,\ n \to +\infty \end{array} \right| = \frac{1}{3} + \lim_{(n)} \frac{-\frac{1}{2}}{1+\infty} + \frac{1}{3} + 0 = \frac{1}{3}!$$

(iznad razlomka: $\nearrow 0$)

Δ $$\lim_{(n)} \left(\frac{n+1}{n-1}\right)^n = \lim_{(n)} \left(\frac{n-1+1+1}{n-1}\right)^n = \lim_{(n)} \left(1 + \frac{2}{n-1}\right)^{n-1+1} =$$

$$= \lim_{(n)} \left(1 + \frac{2}{n-1}\right)^{(n-1)+1} = e^2$$

Δ $\lim_{(n)} (1+\alpha)\sin\frac{n\pi}{2}$, $\sin\frac{n\pi}{2}$ za $n = 2n \to 0$

za $n = 2n-1 \Rightarrow$

$$\Rightarrow \frac{(2n-1)\pi}{2} = \sin\left(n\pi - \frac{\pi}{2}\right) = -\cos n\pi.$$

$$-\cos n\pi = (-1)^n \begin{array}{l} \xrightarrow{\text{za } 2n} -1 \\ \xrightarrow{\text{za } 2n-1} 1 \end{array}$$ } NIZ DIVERGIRA U ŠIREM SMISLU.

Ako stavimo da je $\alpha = -1$ tada ćemo dobiti nulu ispred $\sin\frac{\pi n}{2}$ pa će niz konvergirati ka nuli $\alpha = -1$!

Prema tome je: $\lim_{(n)} a_n = \frac{1}{3} + e^2 + 0 = \frac{1}{3} + e^2$

② Dokazati da je: $\lim_{(n)} \left(1+\frac{1}{n}\right)^n = e = 2{,}71\ldots$

Potrebno je poznavati Bernulijevu nejednakost i Binomni obrazac. Dokažimo da je niz monotono rastući! Prema Bernuliju je:

za $x = -\frac{1}{n^2}$, $n = 2,3,\ldots$, $-1 < x$ $\quad n(1+x)^n \geq 1+nx \Rightarrow$

$\left(1-\frac{1}{n^2}\right)^n > 1 - n\cdot\frac{1}{n^2} \Rightarrow \left(1-\left(\frac{1}{n}\right)^2\right)^n > 1-\frac{1}{n} \Rightarrow \left(1-\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right)^n > 1-\frac{1}{n}$

ovako dobijem kvadrat

prebacujemo

$\underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n} > \frac{1-\frac{1}{n}}{\left(1-\frac{1}{n}\right)^n} = \left(1-\frac{1}{n}\right)^{1-n}$

$\left(1+\frac{1}{n}\right)^n > \left(1-\frac{1}{n}\right)^{1-n} = \left(\frac{n-1}{n}\right)^{1-n} = \left(\frac{n}{n-1}\right)^{n-1} = \left(1+\frac{n}{n-1}-1\right)^{n-1} \Rightarrow$

$\underbrace{\left(1+\frac{1}{n}\right)^n}_{a_n} > \underbrace{\left(1+\frac{1}{n-1}\right)^{n-1}}_{a_{n-1}}$

$a_n > a_{n-1}$ – niz je monotono rastući što smo prikazali preko Bernulijeve nejednakosti.

Pokažimo da je niz ograničen!

$a_n = \left(1+\frac{1}{n}\right)^n = \binom{n}{0} + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \ldots + \binom{n}{k}\frac{1}{n^k} + \ldots + \binom{n}{n}\frac{1}{n^n} =$

$= 1 + 1 + \underbrace{\text{ostatak reda}}_{a_n} = 2 + a_n > 2$ tj. $2 < a_n < ?$

$a_n = 1 + 1 + \frac{n(n-1)}{n^2}\cdot\frac{1}{2!} + \frac{n(n-1)(n-2)}{n^3}\cdot\frac{1}{3!} + \ldots + \frac{n(n-1)\cdot\ldots\cdot(n-(k-1))}{n^k}\cdot\frac{1}{k!} +$

$+ \ldots + \frac{n(n-1)\cdot\ldots\cdot 1}{n^n}\cdot\frac{1}{n!} = 2 + \frac{1}{2!}\underbrace{\left(1-\frac{1}{n}\right)}_{<1} + \frac{1}{3!}\underbrace{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}_{<1} + \ldots$

$+ \ldots + \frac{1}{k!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{k-1}{n}\right) + \ldots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdot\ldots\cdot\left(1-\frac{n-1}{n}\right) <$

$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} < 2 + \underbrace{\frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}}$

$\frac{1}{3!} = \frac{1}{2\cdot 3} < \frac{1}{2\cdot 2}$ $\qquad S_n = 1 + 1\cdot\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} < 1 + \frac{1}{1-\frac{1}{2}} = 3$

tj. $\boxed{2 < a_n < 3}$.

(Z) Ako je $a_n = \frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{n^3-1}{n^3+1}$ dokazati da je $\lim a_n = \frac{2}{3}$.

$$a_n = \frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{n^3-1}{n^3+1} = \prod_{k=2}^{n}\frac{k^3-1}{k^3+1} = \prod_{k=2}^{n}\frac{(k-1)}{(k+1)}\prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1} =$$

$$= \left| \begin{array}{l} \prod_{k=2}^{n}\frac{(k-1)}{(k+1)} = \frac{1}{3}\cdot\frac{2}{4}\cdot\frac{3}{5}\cdots\frac{(n-1)}{(n+1)} = \frac{(n-1)!}{\frac{(n+1)!}{2}} = \frac{2(n-1)!}{(n-1)!(n+1)\cdot n} = \frac{2}{n(n+1)} \\ \prod_{k=2}^{n}\frac{k^2+k+1}{k^2-k+1} = \frac{7}{3}\cdot\frac{13}{7}\cdot\frac{21}{13}\cdots\frac{n^2+n+1}{n^2-n+1} = \frac{n^2+n+1}{3} \end{array} \right| =$$

$$= \frac{2}{n(n+1)}\cdot\frac{(n^2+n+1)}{3} = \frac{2}{3}\cdot\frac{n^2+n+1}{n^2+n} \quad \Big/ \lim_{(n)}$$

$$\lim_{(n)} a_n = \frac{2}{3}.$$

(Z) Ako je $a_1 = \sqrt{2}$, $a_2 = \sqrt{2+\sqrt{2}}, \ldots, a_n = \underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}_{n \text{ korijena}}$

Dokazati da je limes: $\lim_{(n)} a_n = 2!$

$a_1 = \sqrt{2} > 0 \Rightarrow$ intuitivno pretpostavljamo da je $a_2 = \sqrt{2+\sqrt{2}} < \sqrt{4} = 2$ i $a_n > 0$.

$\vdots$

$a_n = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}} < \sqrt{2+\sqrt{2+\ldots+2}} < 2 \Rightarrow$ također je intuitivno i $a_n < 2$.

Sada predpostavljamo: $0 < a_n < 2$

$$a_{n+1} - a_n = \sqrt{2+a_n} - a_n = \frac{\sqrt{2+a_n} - a_n^2}{\sqrt{2+a_n}+a_n} = \frac{(2-a_n)(a_n+1)}{\sqrt{2+a_n}+a_n}$$

kako je $a_n < 2$ to je $2 - a_n > 0$ pa je

$\frac{(2-a_n)(a_n+1)}{\sqrt{2+a_n}+a_n} > 0$ pa je $a_{n+1} > a_n$

$a_{n+1} = \sqrt{2+a_n}$ – Ako $a_n \to \alpha$ tada i $a_{n+1} \to \alpha$ pa je

$\alpha = \sqrt{2+\alpha} \Rightarrow \alpha = 2.$

(Z) $\lim_{(n)} \frac{a^n}{n!} = 0$, $(a>0)$

$\lim_{(n)} \left(\frac{a}{1}\cdot\frac{a}{2}\cdot\frac{a}{3}\cdots\frac{a}{n}\right) = L \;/\cdot \ln$

$\ln L = \lim_{(n)} \left(n\ln a - \ln(n!)\right) \Rightarrow \ln L = \lim_{(n)} \left(\frac{n\ln a - \ln(1\cdot 2\cdot 3\cdots n)}{n}\right)n$

$\ln L = \lim_{(n)} \left(\ln a - \frac{\ln 1 + \ln 2 + \ldots + \ln n}{n}\right)\cdot n$

Uzmimo da je : $\boxed{\frac{\ln 1+\ln 2+\ldots+\ln n}{n} = \alpha_n}$

$\frac{\alpha_n - \alpha_{n-1}}{\beta_n - \beta_{n-1}} = \frac{\ln n}{n-(n-1)} = \frac{\ln n}{1} = \ln n$

$\ln L = \lim_{(n)} \left(\overset{const.}{\ln a} - \overset{\to\infty}{\ln n}\right)\cdot n = -\infty \Rightarrow \underline{\ln L = -\infty} \Rightarrow$

$\log_e L = -\infty \Rightarrow L = e^{-\infty} \Rightarrow L = \frac{1}{e^{\infty}} = 0 \Rightarrow \underline{L=0}$

(Z) Provjeriti da li je : $\lim_{(n)} \sqrt[n]{a} = 1$ za $\underline{a>0}$.

$\lim_{(n)} \sqrt[n]{a} = 1$ za $a>0 \Rightarrow \lim_{(n)} a^{\frac{1}{n}} = 1$ jer $\frac{1}{n}\to 0,\ n\to+\infty$.

ili za $a>1 \Rightarrow \sqrt[n]{a} > 1 \;/^{(n)}$

$a = (1+\sqrt[n]{a}-1)^n > 1 + n(\sqrt[n]{a}-1) > n(\sqrt[n]{a}-1) \Rightarrow$

$a > n(\sqrt[n]{a}-1) \Rightarrow \underline{0 < (\sqrt[n]{a}-1) < \frac{a}{n} < \varepsilon}$

Za $\varepsilon>0$ i $n > \frac{a}{\varepsilon}$ $\sqrt[n]{a} \to$ kad $n\to\infty$.

Za $0<a<1 \Rightarrow \frac{1}{a} > 1$ tada $\sqrt[n]{\frac{1}{a}} \to 1$ kad $n\to+\infty$.

(Z) Dokazati da je $\lim_{(n)} \frac{n}{3^n} = 0$!

$\lim_{(n)} \frac{n}{3^n} = \lim_{(n)} \frac{n}{(1+(3-1))^n} = \lim_{(n)} \frac{n}{1^n + \binom{n}{1}(3-1) + \binom{n}{2}(3-1)^2 + \ldots + \binom{n}{n}(3-1)^n} =$

$= \left| 1^n + \binom{n}{1}(3-1) + \binom{n}{2}(3-1)^2 + \ldots + \binom{n}{n} = \underbrace{1 + n(3-1) + \frac{n(n-1)}{2}(3-1)^2 + \ldots + (3-1)^n}_{>0} > \frac{n(n-1)}{2}(3-1)^2 \right| \Rightarrow$

$$0 < \frac{n}{1^n + \binom{n}{1}(3-1) + \binom{n}{2}(3-1)^2 + \dots + \binom{n}{n}(3-1)^n} < \frac{n}{\frac{n(n-1)}{2}(3-1)^2} =$$

$$= \frac{2}{(n-1)(3-1)^2} \Rightarrow \lim_{(n)} \frac{2}{(n-1)(3-1)^2} = 0 \quad \text{ŠTD.}$$

(Z) $\lim_{(n)} \frac{1\cdot 2 + 2\cdot 3 + \dots + n(n+1)}{n^3}$

$$a_n = \frac{1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \dots + n(n+1)}{n^3} \Rightarrow \begin{array}{l} 1\cdot 2 + 2\cdot 3 + \dots + n(n+1) > 1^2 + 2^2 + \dots + n^2 \\ 1\cdot 2 + 2\cdot 3 + \dots + n(n+1) < 2^2 + 3^2 + \dots + (n+1)^2 \end{array}$$

$$\left.\begin{array}{l} \lim_{(n)} \frac{1^2 + 2^2 + \dots + n^2}{n^3} = \lim_{(n)} \frac{n}{6} \frac{(2n+1)(n+1)}{n^3} = \frac{1}{3} \\ \lim_{(n)} \frac{2^2 + 3^2 + \dots + (n+1)^2}{n^3} = \frac{1}{3} \end{array}\right\} \lim_{(n)} a_n = \frac{1}{3}.$$

(Z) Potrebno je provjeriti sljedeće limese!

1° $\lim_{(n)} a^n = 0$, $(|a| < 1)$;

2° $\lim_{(n)} a^n = \infty$, $(a > 1)$;

3° $\lim_{(n)} a^n = 1$, $(a = 1)$.

RJ: 1° $\lim a^n = 0$, $(|a| < 1)$

kako je $|a| < 1$ to je $|a| = \frac{1}{|b|}$, $|b| > 1 \Rightarrow$ da b pišemo u obliku $b = 1 + h$, $h > 0$.

$|a|^n = \frac{1}{(1+h)^n} \le \frac{1}{1+nh}$; $\frac{1}{1+nh} \to 0$, kad $n \to \infty$.

Tako je i $|a|^n \to 0$, $n \to \infty$

2° $\lim_{(n)} a^n = \infty$; $(a > 1)$

kako je $a > 1$ to a pišemo kao $1+h$, $h > 0$ tj.,

$a = 1 + h$, $h > 0$. $\quad a^n = (1+h)^n \ge 1 + nh$, $h > 0$ $/ \cdot \lim_{(n)}$

$\lim_{(n)} (1 + nh) = \infty$

3° $\lim_{(n)} a^n = 1$ za $a = 1$.

(Z) $\lim\limits_{(n)} \dfrac{a_0 n^k + a_1 n^{k-1} + a_2 n^{k-2} + \dots + a_k}{b_0 n^h + b_1 n^{h-1} + \dots + b_h}$, $(a_0, b_0 \neq 0)$

Odmah se vidi da izraz treba dijeliti sa $n^{\text{(ma neki broj)}}$. Ali imamo k i h eksponente pa imamo tri slučaja.

1° za $k=h$ dijelimo sa $n^k = n^h \Rightarrow \lim \dfrac{a_0 n^k + \dots + a_k}{b_0 n^h + \dots + b_h} = \dfrac{a_0}{b_0}$

2° za $k<h$ djelimo sa n^k pa imamo:

$$\lim_{(n)} \frac{a_0 + \frac{a_1}{n^{h-k}} + \dots + \frac{a_k}{n^h}}{b_0 + \frac{b_1}{n} + \dots + \frac{b_h}{n^h}} = 0$$

3° za $k>h$ djelimo sa $n^k \Rightarrow$

$$\Rightarrow \lim_{(n)} \frac{a_0 n^{k-h} + \dots + \frac{a_k}{n^k}}{b_0 + \frac{b_1}{n} + \dots + \frac{b_h}{n^k}} = \operatorname{sgn}(a_0 b_0) \cdot \infty, \text{ tj. } : \pm\infty.$$

(Z) $\lim\limits_{(n)} \left[\dfrac{1+3+5+\dots+(2n-1)}{n+1} - \dfrac{2n+1}{2} \right] =$

$= \Big| \; 1+3+5+\dots+(2n-1)$ je aritmetički : $\underbrace{a_1 + a_1 + d + a_1 + 2d + \dots + a_1 + nd}_{S_n = \frac{n}{2}(a_1 + a_n)} =$

$= 1 + 1 + 2 + 1 + 2 \cdot 2 + 1 + 2 \cdot 3 + \dots + 1 \cdot 2 \cdot n \Rightarrow a_1 = 1, \; a_n = (2n-1)$ pa je naš red : $1+3+5+\dots+(2n-1) = \frac{n}{2}\left((2n-1)+1\right) = \frac{n}{2}(2n) = n^2 \Big| =$

$$= \lim_{(n)} \left[\frac{n^2}{n+1} - \frac{2n+1}{2} \right] = \lim_{(n)} \left[\frac{2n^2 - (2n+1)(n+1)}{2(n+1)} \right] =$$

$$= \lim_{(n)} \left[\frac{2n^2 - 2n^2 - 2n - n - 1}{2(n+1)} \right] = \lim_{(n)} \left[\frac{-3n-1}{2n+2} \right] =$$

$$= \lim_{(n)} \left[\frac{-3 - \frac{1}{n}}{2 + \frac{2}{n}} \right] = -\frac{3}{2}.$$

② Dokazati da je a) $\lim\limits_{(n)} n^k a^n = 0$ $(|a|<1,\ k\in N)$

b) $\lim\limits_{(n)} \sqrt[n]{n} = 1$

a) $\lim\limits_{(n)} n^k a^n = 0,\ |a|<1,\ k\in N$

$|a|<1$ i ako ga stepenujemo ostat će <1, i bivat će sve manji i manji tako da će $\to 0,\ n\to\infty$. Zato, broj a predstavljamo kao $a=\frac{1}{b}$, $|b|>1$, tada je $|b| = 1+h,\ h>0$. Na osnovu ovoga pišemo:

$$|n^k a^n| = \frac{n^k}{(1+h)^n} = \frac{n^k}{\underbrace{1+nh+\ldots+}_{\text{odbacim}} \frac{n(n-1)\cdot\ldots\cdot(n-k)}{(k+1)!}\cdot h^{k+1} \underbrace{+\ldots}_{\text{odbacim}}} <$$

$$< \frac{n^k (k+1)!}{n(n-1)\cdot\ldots\cdot(n-k)h^{k+1}} = \frac{(k+1)!}{\frac{n(n-1)\cdot\ldots\cdot(n-k)h^{k+1}}{n^k}} =$$

$$= \frac{(k+1)!}{h^{k+1}\cdot(1-\frac{1}{n})(1-\frac{2}{n})\cdot\ldots\cdot(1-\frac{k-1}{n})\cdot(n-k)} \to 0,\ n\to\infty, \text{ pa je}$$

$\lim\limits_{(n)} n^k a^n = 0$

b) $\lim\limits_{(n)} \sqrt[n]{n} = 1$. Neka je $\sqrt[n]{n} = 1+h \Rightarrow (1+h)^n = n > \frac{n(n-1)}{2}h^2 =$

$$= \left|(1+h)^n = 1+nh+\ldots+\frac{n(n-1)\cdot\ldots\cdot(n-k)}{(h+1)!}h^{k+1}+\ldots\right| = \left|(1+h)^n = 1+nh+\binom{n}{2}h^2+\binom{n}{3}h^3+\ldots\right| =$$

$$= \left|(1+h)^n = 1+nh+\frac{n(n-1)}{2}h^2+\ldots\right| = n = (1+h)^n > \frac{n(n-1)}{2}h^2 \Rightarrow$$

$\Rightarrow n > \frac{n(n-1)}{2}h^2 \Rightarrow h^2 < \frac{2}{n-1}$ ili $h < \sqrt{\frac{2}{n-1}}$ za $n\to\infty$,

$\sqrt{\frac{2}{n-1}} = h \to 0$. Dakle, $\sqrt[n]{n} = 1+h$. Sada bi trebalo biti da kada $n\to\infty$, $h\to 0$ i ostaje 1.

$(1+h)^n = (\sqrt[n]{n})^n = n > \frac{n(n-1)}{2}h^2 \Rightarrow h^2 < \frac{2}{n-1}$, kada $n\to\infty$, $h\to 0$.

$$\lim\limits_{(n)} h^2 < \lim\limits_{(n)} \frac{2}{n-1} \Rightarrow h^2 < 0.$$

Ⓩ $\lim_{(n)} (1-\frac{1}{3})(1-\frac{1}{6})\cdot\ldots\cdot\underbrace{\left(1-\frac{1}{\frac{n(n+1)}{2}}\right)}$

ovaj član trebamo rastaviti!

Imamo: $1-\frac{1}{\frac{k(k+1)}{2}} = \frac{(k-1)(k+2)}{k(k+1)}$, $k \geq 2$

Tada: $\lim_{(n)} (1-\frac{1}{3})(1-\frac{1}{6})\cdot\ldots\cdot\left(1-\frac{1}{\frac{n(n+1)}{2}}\right) =$

$= \lim_{(n)} \left(\frac{1\cdot4}{2\cdot3}\cdot\frac{2\cdot5}{3\cdot4}\cdot\frac{3\cdot6}{4\cdot5}\cdot\ldots\cdot\frac{(n-1)(n+2)}{n(n+1)}\right) =$

$= \lim_{(n)} \frac{1}{3}\cdot\frac{n+2}{n} = \underline{\frac{1}{3}}.$

Ⓩ Naći limes: $\lim_{(n)} (1-\frac{1}{2^2})(1-\frac{1}{3^2})\cdot\ldots\cdot\underbrace{(1-\frac{1}{n^2})}$

RASTAVITI!

RJ: $1-\frac{1}{k^2} = \frac{k^2-1}{k^2} = \frac{(k-1)(k+1)}{k^2}$, $k \geq 2$

$\lim_{(n)} \left(\frac{1\cdot3}{2^2}\cdot\frac{2\cdot4}{3^2}\cdot\frac{3\cdot5}{4^2}\cdot\frac{4\cdot6}{5^2}\cdot\ldots\cdot\frac{(n-1)(n+1)}{n^2}\right) =$

$= \lim_{(n)} \frac{1}{2}(1+\frac{1}{n}) = \lim \frac{1}{2} + \lim \frac{1}{2n} \to 0 = \frac{1}{2}$

* $\frac{n}{\sqrt{n^2+n}} < \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}} < \frac{n}{\sqrt{n^2+1}}$. To se može objasniti ovako: Neka je $a_n = \frac{1}{100+1} + \frac{1}{100+2} + \ldots + \frac{1}{100+100}$ tada je $\frac{1}{100+1}$ najveći, a $\frac{1}{100+100}$ najmanji član niza. Prema tome je

$$\underbrace{\frac{1}{100+100}+\frac{1}{100+100}+\ldots+\frac{1}{100+100}}_{100 \text{ članova}} < \underbrace{\frac{1}{100+1}+\frac{1}{100+2}+\ldots+\frac{1}{100+100}}_{100 \text{ članova}} < \underbrace{\frac{1}{100+1}+\frac{1}{100+1}+\ldots+\frac{1}{100+1}}_{100 \text{ članova}}$$

Tj. vrijedi: $\frac{n}{100+100} < a_n < \frac{n}{100+1}$

* Primjena košijevog teorema :

Vidi se (nakon iskustva stečenog prerodom 100 zadataka) da je :

$$\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+p} > \frac{p}{n+p} \Big| \text{za } p=n \Big| \Rightarrow \frac{n}{n+n}=\frac{1}{2} \text{ tj. } > \frac{1}{2},$$

pa se zbog toga odmah na početku traži da je broj ε iz intervala $(0,\frac{1}{2})$ pa ako je $|x_{n+p}-x_n|<\varepsilon$ netočno, tada x_n divergira !

Redovi

Kriteriji za konvergenciju redova

redovi sa pozitivnim članovima

* Da bi red bio ⓚ potrebno je da opšti član teži 0, kad $n \to \infty$, tj, da $a_n \to 0$ kad $n \to \infty$ tj, $\lim\limits_{(n)} a_n \to 0$.

* Opšti koši - Bolzanov kriterij:
Red $\sum\limits_{1}^{\infty} a_n$ je ⓚ akko za $\forall \varepsilon > 0$ $\exists n_0 \in N$, $n = n_0(\varepsilon)$ takav da vrijedi: $|a_{n+1} + a_{n+2} + \ldots + a_{n+p}| < \varepsilon$ za $\forall n \geq n_0$ i $\forall p = 1, 2, 3, \ldots$

* Testovi upoređivanja
Neka su dati redovi $\sum\limits_{1}^{\infty} a_n$ (1) i $\sum\limits_{1}^{\infty} b_n$ (2) tada imamo:

1° Ako je za $n \geq n_0$ ispunjena nejednakost: $0 \leq a_n \leq b_n$ tada iz ⓚⓡ(2) ⇒ ⓚⓡ(1) i iz ⓓⓡ(1) ⇒ ⓓⓡ(2).

2° Ako je $\lim\limits_{(n)} \frac{a_n}{b_n} = k$ $(0 \leq k \leq \infty)$ tada iz ⓚⓡ(1) za $k < \infty$ ⇒ ⓚⓡ(2), a iz ⓓ reda (2), $k > 0$ ⇒ ⓓⓡ(1); kao komparativni red (2) često se upotrebljava red $\sum\limits_{1}^{\infty} \frac{1}{n^p}$ koji ⓚ za $p > 1$ i ⓓ za $p \leq 1$.

▷ * Dalamberov test količnika

Ako je $\overline{\lim\limits_{(n)}} \frac{a_{n+1}}{a_n} = q$, $(a_n > 0)$, tada je za $q < 1$ ⓡ(1)ⓚ.

Ako je $\lim\limits_{(n)} \frac{a_{n+1}}{a_n} = q > 1$ red (1) je ⓓ.

▷ * Rabeov kriterij

Ako je $\lim\limits_{(n)} n\left(\frac{a_n}{a_{n+1}} - 1\right) = p$, $(a_n > 0)$, tada je za $p > 1$ ⓡ(1)ⓚ, a za $p < 1$ ⓓ.

▷ * košijev kriterij korjena

Ako je $\lim\limits_{(n)} \sqrt[n]{a_n} = q$ $(a_n > 0)$, tada je za $q < 1$ ⓡ(1)ⓚ, a za $q > 1$ ⓓ.

▷ * **Logaritamski kriterij**

Neka je $\Delta(\log) = \dfrac{\ln(\frac{1}{a_n})}{\ln n} > 1$ (R) (1) (K), ≤ 1 (R) (1) (D)

Ako je $\lim\limits_{(n)} \Delta(\log) = \lim\limits_{(n)} \dfrac{\ln(\frac{1}{a_n})}{\ln n} > 1$ (R) (K), < 1 (R) (1) (D),

$= 1$ tada nemožemo reći ništa o konvergenciji (R) (1).

* a_n je opšti član reda $\sum\limits_{1}^{\infty} a_n$.

* **Štirlingova formula**

Za dovoljno velike vrijednosti n važi:

$n! = \sqrt{2\pi n} \cdot n^n \cdot e^{-n+\frac{\theta}{12n}}$, $(0 < \theta_n < 1)$ ili $n! = (\frac{n}{e})^n \sqrt{2\pi n}$

tj. $n! = (\frac{n}{e})^n (2\pi n)^{\frac{1}{2}}$.

* **Gausov test**

Neka je $\dfrac{a_n}{a_{n+1}} = \lambda + \dfrac{\mu}{n} + \dfrac{\theta_n}{n^{1+\varepsilon}}$, $(a_n > 0)$ gdje je θ_n ograničena f-ja od n i $\varepsilon > 0$. Tada (R) (1) (K) za $\lambda > 1$ i za $\lambda = 1$ ako je $\mu > 1$ za $\lambda < 1$ i za $\lambda = 1$ ako je $\mu \le 1$ red (1) (D).

* **Cauchyev (kašijev) intedralni test.**

Ako je $f(x)$, $(x > 0)$ nenegativna rastuća f-ja, tada red $\sum\limits_{1}^{\infty} f(n)$ (K) ili (D) istovremeno sa integralom $\int\limits_{1}^{\infty} f(x)\,dx$.

redovi sa proizvoljnim članovima

* **Leibnicov test:**

Naizmjenični red $b_1 - b_2 + b_3 - b_4 + \ldots + (-1)^{n-1} b_n + \ldots$ $(b_n \geq 0)$, tj. $\sum\limits_{1}^{\infty} (-1)^{n-1} b_n$ konvergira ako je $b_n \geq b_{n+1}$ $(n = 1, 2, \ldots)$ i $\lim\limits_{(n)} b_n = 0$. U tom slučaju za ostatak reda:

$R_n = (-1)^n b_{n+1} + (-1)^{n+1} b_{n+2} + \ldots$ važi ocjena:

$R_n = (-1)^n \theta_n b_{n+1}$, $(0 \le \theta_n \le 1)$.

* Abelov test

Red $\sum_{1}^{\infty} a_n b_n$ konvergira ako konvergira red $\sum_{n=1}^{\infty} a_n$ i ako brojevi b_n obrazuju monotono ograničen niz.

* Dirihleov test

Red $\sum_{1n}^{\infty} a_n b_n$ konvergira ako su djelomične sume $A_n = \sum_{1}^{n} a_n$ ograničene i ako b_n monotono teže nuli ($b_n \to 0$) kod $n \to +\infty$.

* Ako je red apsolutno Ⓚ onda je on i konvergentan.

* Bertrandov kriterij

Neka je dat red (1) $\sum_{1}^{\infty} a_n$ i neka je $a_n > 0$ i $n = 1, 2, \ldots$

Tada vrijedi:

I Ako $\exists n_0 \in N$ i $r \in R > 1$ takvi da vrijedi:

$\ln n\left[n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \geq r \quad \forall n \geq n_0$, onda je Ⓡ(1) Ⓚ.

II Ako postoji $n_0 \in N$ takav da vrijedi:

$\ln n\left[\left(n\frac{a_n}{a_{n+1}} - 1\right) - 1\right] \leq 1, \quad \forall n \geq n_0$ tada je red (1) Ⓓ.

važni limesi koji se često koriste kod redova

1. $\lim\limits_{x \to 0} \frac{\sin x}{x} \overset{\circ}{=} 1$

2. $\lim\limits_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$

3. $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e = 2{,}718\ldots$

4. $\lim\limits_{\substack{x \to +\infty \\ \beta > 0 \\ \forall \alpha}} \frac{x^\alpha}{e^{\beta x}} = 0$

5. $\lim\limits_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \ln n\right) = \gamma = 0{,}57\ldots$

6. $\lim\limits_{n \to \infty} \sqrt[n]{n^k} = 1, \quad \forall k \in R$

7. $\lim\limits_{n \to \infty} \sqrt[n]{(\log_a n)^k} = 1, \quad \forall k \in R, \quad 0 < a \neq 1$

8. $\lim\limits_{n \to \infty} (1 + \alpha_n)^{\frac{1}{\alpha_n}} = e$, Ako $\alpha_n \to 0, n \to \infty$ (monotono)

ekvivalencije kada $x \to 0$

$\sin x \sim 0$

$\sin x \sim x$

$\sin x \sim x - \frac{x^3}{6}$

$\sin x \sim x - \frac{x^3}{6} - O(x^5)$

$\operatorname{sh} x \sim 0$

$\operatorname{sh} x \sim x$

$\operatorname{sh} x \sim x + \frac{x^3}{6}$

$\operatorname{sh} x = x + \frac{x^3}{6} + O(x^5)$

$\cos x \sim 1$

$\cos x \sim 1 - \frac{x^2}{2}$

$\cos x \sim 1 - \frac{x^2}{2} + \frac{x^4}{24}$

$\operatorname{ch} x \sim 1$

$\operatorname{ch} x \sim 1 + \frac{x^2}{2}$

$\operatorname{ch} x \sim 1 + \frac{x^2}{2} + \frac{x^4}{24}$

$e^x - 1 \sim 0$

$e^x - 1 \sim x$

$e^x - 1 \sim x + \frac{x^2}{2}$

$e^x = 1 + x + \frac{x^2}{2} + O(x^3)$

$\ln(1+x) \sim 0$

$\ln(1+x) \sim x$

$\ln(1+x) \sim x - \frac{x^2}{2}$

$\ln(1+x) \sim x - \frac{x^2}{2} + O(x^3)$

$(1+x)^\alpha \sim 1$

$(1+x)^\alpha \sim 1 + \alpha x$

$(1+x)^\alpha \sim 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2$

$(1+x)^\alpha \sim 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + O(x^3)$

kada $x \to 1$

$\ln x \sim 1 - x$, $x \to 1$

Smjene:

$2 \sin \frac{x}{2} \cos \frac{x}{2} = \sin x$

$\frac{x}{2} = t$, $t \to 0$

$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$

* Ako je S_n - n-ta parcijalna suma nekog reda $\sum_1^\infty a_n$ tada je suma tog reda $S = \lim_{n \to \infty} S_n$.

(Z) Na osnovu $\lim\limits_{(n)} \left(1+\frac{1}{n}\right)^n = e$ dokazati $\lim\limits_{n\to-\infty} \left(1+\frac{1}{n}\right)^n = e$!

Imamo:

$$\lim_{(-\infty)} \left(1+\frac{1}{n}\right)^n = \begin{vmatrix} n=-m, \\ m=n \end{vmatrix} = \lim_{n\to+\infty} \left(1-\frac{1}{n}\right)^{-n} = \lim_{(n)} \left(\frac{n-1}{n}\right)^{-n} = \lim_{(n)} \left(\frac{n}{n-1}\right)^{n} =$$

$$= \lim_{(n)} \left(1+\frac{1}{n-1}\right)^n = \lim_{(n)} \left(1+\frac{1}{n-1}\right)^{n-1} \cdot \left(1+\frac{1}{n-1}\right) = \lim_{(n)} \left(1+\frac{1}{n-1}\right)^{n-1} \cdot 1 = \underline{e}.$$

(Z) Pokazati da $\forall x \in \mathbb{R}\setminus\{0\}$ vrijedi $\lim\limits_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e$!

Izraz je tipa 1^∞ pa je neodređen! Znamo da se svaki realni broj nalazi između dva prirodna broja npr: n i $n+1$. Dakle, imamo: $0<n<x<n+1$, Sada je odavde: $\frac{1}{n+1} < \frac{1}{x} < \frac{1}{n}$. Svim stranama ovog izraza dodamo 1:

$\left(1+\frac{1}{n+1}\right) < 1+\frac{1}{x} < \frac{1}{n}+1$. Stepenujemo sa n, x i $n+1$ respektivno članove $1+\frac{1}{n+1}$, $1+\frac{1}{x}$, $1+\frac{1}{n}$ i uzmemo $\lim$ u obzir: $n<x<n+1$ imamo odavde:

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^{n+1}, \text{ Odavde:}$$

$$\left(1+\frac{1}{n+1}\right)^{n+1} \cdot \left(1+\frac{1}{n+1}\right)^{-1} < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^n \cdot \left(1+\frac{1}{n}\right) \Big/ \circ \lim_{(n)} \Rightarrow$$

$$e < \lim_{(n)} \left(1+\frac{1}{x}\right)^x < e \text{ tj. } \underline{\lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = e}.$$

(Z) Pokazati da je $\lim\limits_{x\to0+} (1+x)^{\frac{1}{x}} = e$ i $\lim\limits_{x\to0-} (1+x)^{\frac{1}{x}} = e$.

$$L_1 = \lim_{x\to+\infty} \left(1+\frac{1}{x}\right)^x = \begin{vmatrix} y=\frac{1}{x}, y\to0_+ \\ x=y \end{vmatrix} = \lim_{x\to0+} (1+x)^{\frac{1}{x}} = e,$$

$$L_2 = \lim_{x\to-\infty} \left(1+\frac{1}{x}\right)^x = \begin{vmatrix} y=\frac{1}{x}, y\to0_- \\ x\equiv y \end{vmatrix} = \lim_{x\to0-} (1+x)^{\frac{1}{x}} = e$$

Dakle,

$$\underline{\lim_{x\to0} (1+x)^{\frac{1}{x}} = e}.$$

(Z) Dokazati da je $\lim_{x\to 0}\frac{\ln(1+x)}{x}=1$!

Iz prethodnog zadatka imamo da je $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$ /ln

$\Rightarrow \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}}=\ln e=1, \quad \lim_{x\to 0}\frac{\ln(1+x)}{x}=1$

(Z) Dokazati da je $\lim_{x\to 0}\frac{e^x-1}{x}=1$!

kako je ovo neodređen izraz tipa $\frac{0}{0}$ moramo uraditi regularnu transformaciju.

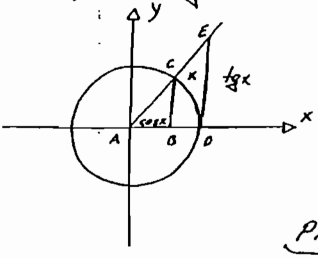
$$\lim_{x\to 0}\frac{\ln(1+x)}{x}=1 \Rightarrow \left|\begin{matrix}1+x=e^t, & t\to 0,\\ t\equiv x & \end{matrix}\right| = \lim_{x\to 0}\frac{\ln e^x}{e^x-1}=1.$$

Očito je da ako je $\lim\frac{a}{b}=1$ onda je i $\lim\frac{b}{a}=1$

pa je: $\lim_{x\to 0}\frac{\ln e^x}{e^x-1}=\lim_{x\to 0}\frac{e^x-1}{\ln e^x}=1$

(Z) Dokazati $\lim_{x\to 0}\frac{\sin x}{x}=1$!

Izraz tipa $\frac{0}{0}$. Ovaj izraz ima simetrično definiciono područje pa je $L_1=L_2$, pa da bi smo pokazali da $\frac{\sin x}{x}\to 1, x\to 0$ potrebno je izvršiti regularnu transformaciju. U tom cilju posmatrajmo sliku:



Površine trouglova su:
ΔABC,
ΔADE, te isječak $\measuredangle ADC$, a oni stoje u odnosu:

$$\underbrace{P_{ABC} < P_{isječka} < P_{ADE}}$$

$$0<\frac{\sin x\cos x}{2}<\frac{x}{2}<\frac{\operatorname{tg} x}{2} \quad \Big/\cdot\frac{2}{\sin x}$$

$$\Rightarrow \quad 0<\cos x<\frac{x}{\sin x}<\frac{1}{\cos x} \Rightarrow$$

$$\Rightarrow \quad \frac{1}{\cos x}>\frac{\sin x}{x}>\cos x>0 \Rightarrow$$

$$\Rightarrow \quad 0<\cos x<\frac{\sin x}{x}<\frac{1}{\cos x} \quad \Big/\cdot\lim_{x\to 0},$$

$$0<\cos x<\frac{\sin x}{x}<\frac{1}{\cos x} \quad \Big/ \cdot \lim_{x\to 0}$$

$$0<1<\frac{\sin x}{x}<1 \quad \Rightarrow \quad \left.\begin{matrix} L_1=1 \\ L_2=1 \end{matrix}\right\} \Rightarrow \underline{L=1}.$$

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Odavde vrijede i sljedeći limesi: $\lim_{x\to 0}\frac{\operatorname{tg} x}{x}=1$

$$\lim_{x\to 0}\frac{\arcsin x}{x}=1$$

$$\lim_{x\to 0}\frac{\operatorname{arctg} x}{x}=1$$

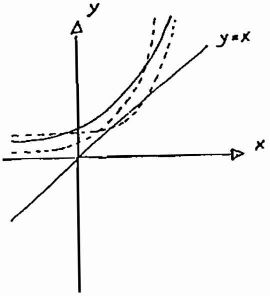
(Z) Dokazati $L=\lim_{x\to+\infty}\frac{x^\alpha}{e^{\beta x}}=0,\ \beta>0,\ \forall\alpha!$

1° $\alpha<0$ $\quad L=\lim_{x\to+\infty}\frac{1}{e^{\beta x}\cdot x^{\alpha}}=0,\ \beta>0,\ \forall\alpha<0$

2° $\alpha=0$, $\quad L=\lim_{x\to+\infty}\frac{1}{e^{\beta x}}=0$

3° $\alpha>0$, $\quad L=\frac{x^\alpha}{e^{\beta x}} \overset{\infty/\infty}{=}$ potrebna regularna transformacija.

Primijetimo da je $e^{\varepsilon x}\geq x$ za $x>q$, $x>>$, $\varepsilon>0$ (po volji), a što se vidi sa slike:



Čak, i ako je neka e kriva manja od $y=x$, ona će biti veća za dovoljno velike x.

To znači $e^{\varepsilon x}\geq x$ i možemo pisati

$$\varepsilon=\frac{\beta}{\alpha}$$

$0\leq x\leq e^{\frac{\beta}{\alpha}x} \Big/ : e^{\frac{\beta}{\alpha}x}$ i stepenujemo sa α pa dobijemo:

$0<\frac{x^\alpha}{e^{\beta x}}\leq 1$ znači $0\leq L\leq 1$.

Iz $L=\lim_{x\to+\infty}\frac{x^\alpha}{e^{\beta x}}=\lim_{x\to+\infty}\frac{(x+1)^\alpha\cdot x^\alpha\cdot e^\beta}{e^{\beta(x+1)}(x+1)^\alpha}=\lim_{x\to+\infty}\frac{(x+1)^\alpha}{e^{\beta(x+1)}}\cdot\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{\alpha} e^{\beta}=$

$=\underline{L e^\beta}.$

Dakle, $L = L \cdot e^{\beta}$ $/\!:\ \beta>0 \Rightarrow e^{\beta}>0$ i $e^{\beta}\neq 1 \Rightarrow L \neq const. \neq 0$
$L = +\infty$ – nemoguće jer je $0 \le L \le 1$ pa ostaje $L = 0$.

Dakle, $\forall \alpha$ i $\beta > 0$ imamo: $L = \dfrac{x^{\alpha}}{e^{\beta x}} \longrightarrow 0$, kad $x \to +\infty$.

(Z) Naći $L = \lim\limits_{x\to 0} \dfrac{x - \sin x}{x^3}$.

Kako je izraz neodređen, tipa $\frac{0}{0}$ treba napraviti regularnu transformaciju:

$$L = \lim_{x\to 0} \frac{x - 2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{x^3} \overset{0}{\underset{0}{=}} \left| \begin{matrix} \frac{x}{2} = t, \\ t\to 0 \end{matrix} \right| = \lim_{t\to 0} \frac{2t - 2\sin t\cdot\cos t}{8t^3} = \left| t \equiv x \right| =$$

$$= \frac{1}{4}\lim_{x\to 0}\left(\frac{x - \sin x\cos x}{2x^3}\right) = \frac{1}{4}\lim_{x\to 0}\left(\frac{x - \sin x}{x^3} + \frac{\sin x - \sin x\cos x}{x^3}\right) =$$

$$= \frac{1}{4}\lim_{x\to 0}\left(\frac{x-\sin x}{x^3} + \frac{\sin x}{x}\left(\frac{1-\cos x}{x^2}\right)\right) = \frac{1}{4}\lim_{x\to 0}\frac{x - \sin x}{x^3} +$$

$$+ \frac{1}{4}\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{2\sin^2\frac{x}{2}}{\frac{x^2}{4}} = \frac{1}{4}L + \frac{1}{4}\cdot 1\cdot\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{\left(\frac{x}{2}\right)^2}\cdot\frac{1}{4} =$$

$$= \frac{1}{4}L + \frac{1}{8} \Rightarrow \qquad L = \frac{1}{4}L + \frac{1}{8} \quad\Rightarrow\quad L = \frac{1}{6}$$

$$\lim_{x\to 0}\frac{x - \sin x}{x^3} = \underline{\frac{1}{6}}.$$

Ako je $\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)} = const. = \lambda$ onda je i $f(x) \sim \lambda g(x)$. U svim graničnim procesima u operaciji množenja i dijeljenja i stepenovanja gdje se pojavljuje $f(x)$, $x \to a$, moguće je pod limesom izvršiti direktnu zamjenu $f(x)$ sa $\lambda(g(x))$.

(Z) Riješiti $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}$!

Izraz je $\frac{0}{0}$ pa moramo izvršiti regularnu transformaciju:

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2} = \left|\sin^2\left(\frac{x}{2}\right) \approx \left(\frac{x}{2}\right)^2\right| = \lim_{x\to 0}\frac{2\cdot x^2}{4x^2} = \underline{\frac{1}{2}}.$$

(Z) Naći Limes $\lim_{x\to 0}\frac{\sqrt[7]{1+x\sin x}-1}{\ln^2(1+\operatorname{tg}x)}\ \frac{0}{0}\Rightarrow$ regularna. transf. $\Rightarrow$

$= \lim_{x\to 0}\frac{(1+\sin x)^{\frac{1}{7}}-1}{\ln^2(1+\operatorname{tg}x)} = \lim_{x\to 0}\frac{e^{\ln\frac{1}{7}(1+\sin x)}-1}{\ln^2(1+\operatorname{tg}x)} = \left|\begin{matrix} e^u-1\sim u, \\ u\to 0 \\ \ln(1+u)\sim u, \\ u\to 0\end{matrix}\right\}\ u\equiv x\Bigg| =$

$= \lim_{x\to 0}\frac{\frac{1}{7}\ln(1+x\sin x)}{\operatorname{tg}^2 x} = \left|\begin{matrix}\operatorname{tg}u\sim u,\ u\to 0 \\ \ln(1+u)\sim u \\ u\equiv x\end{matrix}\right| = \lim_{x\to 0}\frac{\frac{1}{7}x\sin x}{x^2} = \frac{1}{7}\lim_{x\to 0}\frac{\sin x}{x} = \underline{\frac{1}{7}}.$

(Z) Naći $\lim_{u\to+\infty}\frac{\ln^2 u}{u} = \left|\begin{matrix}\ln u = x, \\ x\to+\infty \\ u = e^x\end{matrix}\right| = \lim_{x\to+\infty}\frac{x^2}{e^x} = \underline{0}.$

Pozitivni redovi - konvergencija sume

(Z) Naći sumu reda (1) $\sum_{n=1}^{\infty}\ln(1+\frac{1}{n})$.

Ako sumu reda označimo sa S tada je $S=\lim_{n\to+\infty}S_n$, a S_n je n-ta parcijalna suma reda!

$S_n = \sum_{k=1}^{n}\ln(1+\frac{1}{k}) = \sum_{k=1}^{n}\ln\left(\frac{k+1}{k}\right) = \sum_{k=1}^{n}\ln(k+1) - \sum_{k=1}^{n}\ln k =$

$= \left|\sum_{1}^{n}\ln(k+1) = \ln(1+1)+\ln(1+2)+\ldots+\ln(k+1) = \ln 2+\ln 3+\ldots+\ln(k+1) = \sum_{2}^{n+1}\ln k\right| =$

$= \ln 2+\ln 3+\ldots+\ln n+\ln(n+1) - (\ln 1+\ln 2+\ldots+\ln n) =$

$= \ln(n+1) - \underbrace{\ln 1}_{0} = \underline{\ln(n+1)}..$

$S_n = \ln(n+1) \Rightarrow \lim_{(n)}\ln(n+1) = \underline{\infty}.$

(Z) Naći sljedeće Limese:

1° $L_1 = \lim_{x\to 0}\frac{\operatorname{sh}x}{x}$

Znamo da je $\lim_{x\to 0}\frac{e^x-1}{x}=1$, po definiciji hiperbolnog sinusa je: $\operatorname{sh} = \frac{e^x-e^{-x}}{2}$; hiperbolni kosinus $\operatorname{ch}x = \frac{e^x+e^{-x}}{2}$

imamo da je: $L_1 = \lim_{x\to 0}\frac{e^x-e^{-x}}{2x}\Big/\cdot\frac{e^x}{e^x} = \lim_{x\to 0}\frac{e^{2x}-1}{2x}\cdot\frac{1}{e^x} = \underline{1}.$

2° $L_2 = \lim_{x\to 0} \frac{\sin x}{x}$ preko $\lim_{x\to 0} \frac{\operatorname{sh} x}{x}$.

Uzmimo $L_1 = \lim_{x\to 0} \frac{\operatorname{sh} x}{x} = \lim_{x\to 0} \frac{e^x - e^{-x}}{2x}$. Ovdje ide smjena:

$|x \equiv iy| \Rightarrow L_1 = \lim_{x\to 0} \frac{e^{iy} - e^{-iy}}{2iy}$. Prema definiciji kompleksnog

broja imamo: $e^{iy} = \cos y + i\sin y$

$e^{-iy} = \cos y - i\sin y$ što imamo da je:

$$L_1 = \lim_{x\to 0} \frac{\cos y + i\sin y - \cos y + i\sin y}{2iy} = \lim_{x\to 0} \frac{2i\sin y}{2iy} =$$

$$= \lim_{x\to 0} \frac{\sin y}{y} = |y \equiv x| = \lim_{x\to 0} \frac{\sin x}{x} = \underline{1}.$$

3° $L_3 = \lim_{x\to 0} \frac{\operatorname{sh} x - x}{x^3}$.

Uzmimo da je $x = 2y$, $x\to 0 \Rightarrow y \to 0 \Rightarrow L_3 = \lim_{y\to 0} \frac{\operatorname{sh} 2y - 2y}{8y^3} =$

$$= \lim_{y\to 0} \frac{2\operatorname{sh} y \operatorname{ch} y - 2y}{8y^3} = \frac{1}{4} \lim_{y\to 0} \frac{\operatorname{sh} y \operatorname{ch} y - y}{y^3} =$$

$$= \frac{1}{4} \lim_{y\to 0} \frac{\operatorname{sh} y + \operatorname{sh} y \operatorname{ch} y - \operatorname{sh} y - y}{y^3} = \frac{1}{4} \lim_{y\to 0} \left(\frac{\operatorname{sh} y - y}{y^3} + \frac{(\operatorname{ch} y - 1)\operatorname{sh} y}{y^3} \right) =$$

$$= |y \equiv x| = \frac{1}{4} \lim_{x\to 0} \left(\frac{\operatorname{sh} x - x}{x^3} + \frac{(\operatorname{ch} x - 1)\operatorname{sh} x}{x^3} \right) \Rightarrow$$

$$L = \frac{1}{4}\Bigg(L + \underbrace{\lim_{x\to 0} \left(\frac{(\operatorname{ch} x - 1)\operatorname{sh} x}{x^3} \right)}\Bigg) = \frac{1}{4}L + \frac{1}{8} \Rightarrow \underline{L_3 = \frac{1}{6}}..$$

$$\lim_{x\to 0} \underbrace{\frac{\operatorname{sh} x}{x}}_{\to 1} \left(\frac{\operatorname{ch} x - 1}{x^2} \right) = \lim_{x\to 0} \frac{2\operatorname{sh}^2 \frac{x}{2}}{\frac{x^2}{4}} = \frac{1}{2}$$

4° $L_4 = \lim_{x\to 0} \frac{e^x - 1 - x}{x^2}$.

Kako je $\operatorname{sh} x + \operatorname{ch} x = \frac{e^x - e^{-x}}{2} + \frac{e^x + e^{-x}}{2} = \underline{e^x}..,$ što imamo:

$$L_4 = \lim_{x\to 0} \frac{\operatorname{sh} x + \operatorname{ch} x - 1 - x}{x^2} = \lim_{x\to 0} \Bigg(\underbrace{\frac{\operatorname{sh} x - x}{x^3} \cdot x}_{\frac{1}{6}\cdot x = 0} + \frac{\operatorname{ch} x - 1}{x^2} \Bigg) =$$

$$= \lim_{x\to 0} \frac{2\operatorname{sh}^2 \frac{x}{2}}{\frac{x^2}{4}} \cdot \frac{1}{4} = \frac{1}{2} \quad \text{pa je:} \quad \lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \underline{\frac{1}{2}}.$$

5° $L_5 = \lim\limits_{x\to 0} \dfrac{\ln(1+x)-x}{x^2}$

Stavimo li smjenu: $|1+x=e^t,\ t\to 0|$ imamo:

$L_5 = \lim\limits_{t\to 0} \dfrac{t-e^t+1}{(e^t-1)^2} = \left| \begin{matrix} t\equiv x \\ x\to 0 \end{matrix} \right| = -\lim\limits_{x\to 0} \dfrac{e^x-1-x}{(e^x-1)^2}$. Nazivnik možemo

napisati u obliku: $\left(\dfrac{e^x-1}{x}\right)^2 \cdot x^2$ a znamo da je $\dfrac{e^x-1}{x} \to 1,\ x\to 0$

pa je: $L_5 = -\lim\limits_{x\to 0} \dfrac{e^x-1-x}{x^2} = -\dfrac{1}{2}$

$$L_5 = \lim_{x\to 0} \frac{\ln(1+x)-x}{x^2} = -\frac{1}{2}. \qquad \boxed{\lim_{x\to 0} \frac{\operatorname{sh}x+x}{x} = 2.}$$

6° $L_6 = \lim\limits_{x\to 0} \dfrac{\operatorname{ch}x-1-\frac{x^2}{2}}{x^4} = \left| \begin{matrix} \frac{x}{2}=t \\ t\to 0 \end{matrix},\ \begin{matrix} t\equiv x \\ x\to 0 \end{matrix} \right| \Rightarrow$

$L_6 = \lim\limits_{x\to 0} \dfrac{\operatorname{sh}^2x-x^2}{x^4}$ tj. $\lim\limits_{x\to 0} \dfrac{(\operatorname{sh}x-x)(\operatorname{sh}x+x)}{x^4} \cdot \dfrac{1}{8} = L_6 =$

$= \dfrac{1}{8} \lim\limits_{x\to 0} \dfrac{\operatorname{sh}x-x}{x^3} \cdot \dfrac{\operatorname{sh}x+x}{x} = \dfrac{1}{8}\cdot\dfrac{1}{6}\cdot 2 = \dfrac{1}{24}$

$$L_6 = \lim_{x\to 0} \frac{\operatorname{ch}x-1-\frac{x^2}{2}}{x^4} = \underline{\frac{1}{24}}.$$

7° $L_7 = \lim\limits_{x\to 0} \dfrac{(1+x)^\alpha-1-\alpha x}{x^2} = |1+x=e^t,\ t\to 0| = \lim\limits_{t\to 0} \dfrac{e^{t\alpha}-1-\alpha(e^t-1)}{(e^t-1)^2} =$

$= \left| \begin{matrix} \text{stavimo u} \\ \text{nazivnik} \end{matrix} \right| = \left|\left(\dfrac{e^t-1}{t}\right)^2 \cdot t^2\right| \Rightarrow \lim\limits_{t\to 0} \dfrac{e^{t\alpha}-1-\alpha(e^t-1)}{\underbrace{\left(\frac{e^t-1}{t}\right)^2}_{\to 1} \cdot t^2}$ tj.

$= \lim\limits_{t\to 0} \dfrac{e^{t\alpha}-1-\alpha(e^t-1)}{t^2} = \left| \begin{matrix} \alpha t = x \\ t = \frac{x}{\alpha} \end{matrix} \right| = \lim\limits_{x\to 0} \dfrac{(e^x-1-\alpha e^{\frac{x}{\alpha}}+\alpha)\cdot\alpha^2}{x^2} =$

$= \lim\limits_{x\to 0} \left(\dfrac{(e^x-1-x)}{x^2} \nearrow \tfrac{1}{2} + \dfrac{x+\alpha-\alpha e^{\frac{x}{\alpha}}}{x^2} \right)\cdot \alpha^2 = \left| \begin{matrix} \frac{x}{\alpha}=t\to 0 \\ x=\alpha t \end{matrix} \right| =$

$= \dfrac{\alpha^2}{2} + \alpha^2 \cdot \lim\limits_{t\to 0} \dfrac{t\alpha+\alpha-\alpha e}{(t\alpha)^2} = \dfrac{\alpha^2}{2} + \dfrac{\alpha^2}{\alpha^2}\cdot\alpha \lim\limits_{t\to 0} \left(\dfrac{e^t-1-t}{t^2}\right) = \left| \begin{matrix} t\equiv x \\ x\to 0 \end{matrix} \right| =$

$= \dfrac{\alpha^2}{2} - \dfrac{\alpha}{2} = \dfrac{\alpha(\alpha-1)}{2} = \underline{\dbinom{\alpha}{2}}.$

(Z) Ispitati konvergenciju reda $\sum_{1}^{\infty} \sin n\alpha$, $0<\alpha<\pi$.

Pretpostavimo da dati red konvergira, u tom slučaju mu opšti član mora $\to 0$, tj. $\sin n\alpha \to 0$, $n\to\infty$. Ali tada, po Bolzanovom stavu i $\lim_{(n)} \sin(n+1)\alpha \to 0$.

$\sin(n\alpha+\alpha) = \sin n\alpha \cos\alpha + \cos n\alpha \sin\alpha$ pa je

$$\lim_{(n)} (\sin(n+1)\alpha) = \lim_{(n)} (\sin n\alpha \cos\alpha + \cos n\alpha \sin\alpha) = 0$$

$$\lim_{(n)} \underbrace{\sin n\alpha \cos\alpha}_{\to 0} + \lim_{(n)} \cos n\alpha \sin\alpha = \lim_{(n)} \cos n\alpha \cdot \sin\alpha =$$

$= \sin\alpha \cdot \lim_{(n)} (\cos n\alpha)$. kako je $\sin\alpha \neq 0$ jer $0<\alpha<\pi$

to mora da $\lim_{(n)} \cos n\alpha = 0$, ali mi znamo da je

$\sin^2 n\alpha + \cos^2 n\alpha = 1 \quad / \cdot \lim$

$0 + 0 = 1 \Rightarrow$ a to je nemoguće pa pretpostavka da je $\lim_{(n)} \sin n\alpha = \lim_{(n)} a_n = 0$ je (L).

Prema tome nije zadovoljen ni potreban uslov konvergencije pa ovaj red divergira!

(Z) Naći $L = \lim_{(n)} \frac{a^n}{n!}$

Formirajmo red $\sum_{n=1}^{\infty} \frac{a^n}{n!}$. koristeći D'Alamberov kriterij ispitajmo konvergenciju reda:

$a_{n+1} = \frac{a^{n+1}}{(n+1)!}$, $a_n = \frac{a^n}{n!} \Rightarrow \frac{a_{n+1}}{a_n} = \frac{a}{n+1}$

$\lim_{(n)} \frac{a_{n+1}}{a_n} = \lim_{(n)} \frac{a}{n+1} = 0 < 1$ pa red konvergira (k).

(Z) Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{\cos nx - \cos(n+1)x}{n}$, $x\in\mathbb{R}$!

koristimo Dirihleov kriterij koji glasi:

Niz $\sum_{1}^{\infty} a_n b_n$ je (k), ako $b_n \to 0$ monotono opadajući, i ako je a_n ograničen niz tj. ako vrijedi $\left|\sum_{k=1}^{n} a_k\right| < M$.

Uočimo redove (1) $\sum_{k=1}^{\infty} (\cos kx - \cos(k+1)x)$, (2) $(b_n = \frac{1}{n})$.

Očito je zadovoljen uslov: $b_n \to 0$, $n \to \infty$. Pogledajmo da li je niz parcijalnih suma reda (1) ograničen. To je niz:

$$\sum_{k=1}^{n}(\cos kx - \cos(k+1)x) = \sum_{1}^{n}\cos kx - \sum_{1}^{n}\cos(k+1)x = \left|\begin{matrix}\text{navedimo ih}\\ \text{da budu isti}\end{matrix}\right| =$$

$$= \sum_{1}^{n}\cos kx - \sum_{k=2}^{n+1}\cos kx = \cos x + \sum_{k=2}^{n}\cos kx - \sum_{k=2}^{n}\cos kx - \cos(k+1)x =$$

$= \cos x - \cos(k+1)x$. Dakle, imamo da je:

$\left|\sum_{1}^{n} a_n\right| = |\cos x - \cos(k+1)x|$. Međutim znamo da je:

$|\cos x| < 1$ i $|\cos(k+1)x| < 1$ pa je

$\left|\sum_{1}^{n} a_k\right| = |\cos x - \cos(k+1)x| < |\cos x| + |\cos(k+1)x| < 2$ tj., niz je ograničen i konvergira.

Ⓩ Ispitati Ⓚ reda: (1) $\sum_{1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n}}$

$$a_n = \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n}} = \frac{1}{\sqrt[3]{n}\,(\sqrt{n+1}+\sqrt{n})} \to 0$$ -ovo je potreban uslov ali ne i dovoljan.

$$a_n = \frac{1}{\sqrt[3]{n}\cdot\sqrt{n}\left(\sqrt{1+\frac{1}{\sqrt{n}}}+1\right)} = \frac{1}{\sqrt[6]{n^5}\left(1+\sqrt{1+\frac{1}{\sqrt{n}}}\right)}$$

$$\lim_{n\to+\infty}\frac{1}{\sqrt[6]{n^5}\left(1+\sqrt{1+\frac{1}{\sqrt{n}}}\right)} \sim \frac{1}{2n^{5/6}} \sim \frac{1}{n^{5/6}}.$$ Red (1) je ekvivalentan redu $\sum_{1}^{\infty}\frac{1}{n^{5/6}}$. Uzmimo sada log. kriterij!

$$\Delta(\log) = \frac{\ln\frac{1}{a_n}}{\ln n} = \frac{\frac{5}{6}\ln n}{\ln n} = \frac{5}{6} < 1$$ pa red divergira!

Ⓩ Ispitati konvergenciju reda (1) $\sum_{n=3}^{\infty}\frac{1}{(\ln\ln n)^{\ln n}}$! ($a_n > 0$)

(1) $\sum_{n=3}^{\infty}\frac{1}{(\ln\ln n)^{\ln n}}$ (>0) - kakav god bio eksponent, ako je baza pozitivna stepen je pozitivan.

pošto je $a_n > 0$ upotrijebit ćemo logaritamski kriterij!

* $y = \ln n \Rightarrow \log_a b = x \Rightarrow a^x = b \Rightarrow e^y = n$

$\Delta(\log) = \dfrac{\ln\left(\frac{1}{a_n}\right)}{\ln n} = \dfrac{\ln n \cdot \ln\ln\ln n}{\ln n} = \ln\ln n$

$\lim\limits_{(n)} \ln\ln n \to +\infty > 1$ pa ovaj red konvergira!

(Z) Ispitati konvergenciju reda: $\sum\limits_{n=1}^{\infty} \frac{1}{n!}\left(\frac{n}{e}\right)^n$

Upotrijebit ćemo D'Alamberov kriterij:

$\lim\limits_{(n)} \dfrac{a_{n+1}}{a_n} = \dfrac{\dfrac{(n+1)^n}{n!\cdot e^n e}}{\dfrac{n^n}{n!\cdot e^n}} = \lim\limits_{(n)} \dfrac{(n+1)^n}{n^n e} = \lim\limits_{(n)} \left(\dfrac{n+1}{n}\right)^n \dfrac{1}{e} =$

$= \lim\limits_{(n)} \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{e} \geq 1$ pa red (D), jer $\left(1+\frac{1}{n}\right)^n \to e,\ n \to +\infty$

neopadajući. $\left(1+\frac{1}{n}\right)^n > e$.

Preko Štirlingove formule je: $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ pa je

$a_n \sim \frac{1}{n!}\left(\frac{n}{e}\right)^n \sim \frac{1}{\sqrt{2\pi n}} \cdot \frac{\left(\frac{n}{e}\right)^n}{\left(\frac{n}{e}\right)^n} = \frac{1}{\sqrt{2\pi n}} \sim \frac{1}{n^{1/2}}$

$\Delta(\log) = \dfrac{\ln\left(\frac{1}{a_n}\right)}{\ln n} = \dfrac{\frac{1}{2}\ln n}{\ln n} = \frac{1}{2} < 1$ pa red (D).

(Z) Ispitati konvergenciju reda (1) $\sum\limits_{1}^{\infty} \dfrac{1}{(\ln n)^{\ln\ln n}}$.

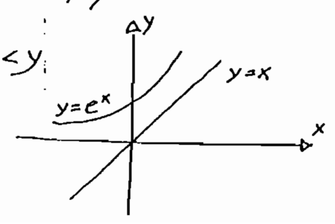
Probajmo kriterijem upoređivanja. Pretpostavimo da red (K). Tada iz napomenutog kriterija i iz konvergencije reda: $\sum\limits_{1}^{\infty} \frac{1}{n^2}$ imamo da je:

$\dfrac{1}{(\ln n)^{\ln\ln n}} < \dfrac{1}{n^2}$ tj., $\Rightarrow n^2 < (\ln n)^{\ln\ln n}$ / o ln

$2\ln n < \ln\ln n \ln\ln n$, Stavimo $\ln n \equiv x,\ x \gg,\ n \gg$

$2x < (\ln x)^2$, Sada stavimo $\ln x = y,\ y \gg \Rightarrow 2\cdot e^y < y^2$ tj.,

$2e^{\frac{y}{2}} < y$



pa je očito da je: $2e^{\frac{y}{2}} \not< y$ (⊥) pa je pretpostavka da red konvergira netačna. Opet ništa ne možemo

pretpostavimo da red divergira i uzmimo u tom cilju red koji divergira! Npr:

$$\frac{1}{(\ln n)^{\ln\ln n}} > \frac{1}{n} \text{ tj. } n > (\ln n)^{\ln\ln n} \; / \circ \ln n \Rightarrow$$

$$\ln n > (\ln\ln n)^2 \;,\; \ln n = x,\; x \gg, n \gg$$
$$x > (\ln x)^2$$
$$e^y > y^2 \Rightarrow e^{y/2} > y \text{ (T) za } y \gg \text{ pa red (1) (D)}$$

(Z) Ispitati konvergenciju reda: $\sum_{1}^{\infty} \frac{(2n-1)!!}{2n!!} \cdot \frac{1}{2^{n+1}}$

Pokušajmo najprije ispitati ovaj red D'Alomberovim kriterijem količnika:

$$a_n = \frac{(2n-1)!!}{2n!!} \cdot \frac{1}{2^{n+1}} \Rightarrow a_{n+1} = \frac{(2n-1)!! \cdot (2n+1)}{2n!! \cdot (2n+2)} \cdot \frac{1}{2^{n+2}} \Rightarrow$$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n-1)!! \cdot (2n+1)}{2n!!(2n+2)} \cdot \frac{1}{2^{n+1} 2}}{\frac{(2n-1)!!}{2n!!} \cdot \frac{1}{2^{n+1}}} = \frac{2n+1}{(2n+2) \cdot 2} = \frac{2n+1+1-1}{2n+2} \cdot \frac{1}{2} =$$

$$= \left(1 - \frac{1}{2n+2}\right) \cdot \frac{1}{2} = \left(\frac{1}{2} - \frac{1}{4n+4}\right)$$

$$\lim_{(n)} \frac{a_{n+1}}{a_n} = \lim_{(n)} \left(\frac{1}{2} - \frac{1}{4n+4}\right) = \underline{\frac{1}{2} < 1}. \text{ pa je red (K)}$$

II način – Štirlingov obrazac

$$n! = \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n},\; a_n > 0,\; a_n = \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot 2n-1}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \cdot \frac{1}{2^{n+1}} =$$

$$= \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 2n-1 \cdot 2n}{2n!! \cdot 2n!! \cdot 2^{n+1}} = \frac{2n!}{2^n n! \, 2^n n! \cdot 2^{n+1}} = \frac{2n!}{4^n n!^2} \cdot \frac{1}{2^{n+1}} =$$

$$= \frac{2n!}{4^n (n!)^2} \cdot \frac{1}{2^{n+1}} \cong \frac{\left(\frac{2n}{e}\right)^{2n} \cdot \sqrt{2\pi 2n}}{4^n \left(2\pi n \left(\frac{n}{e}\right)^{2n}\right) \cdot 2^{n+1}} = \frac{2 \cdot \left(\frac{2n}{e}\right)^{2n} \sqrt{\pi n}}{4^n \left(2\pi n \left(\frac{n}{e}\right)^{2n}\right) \cdot 2^{n+1}} =$$

$$= \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{\pi n}}{4^n \pi n \cdot \left(\frac{n}{e}\right)^{2n} \cdot 2^{n+1}} = \frac{2^{2n} \sqrt{n\pi}}{2^{2n} \pi n \cdot 2^{n+1}} = \frac{\sqrt{n\pi}}{\pi n \cdot 2^{n+1}} = \frac{\sqrt{n\pi}}{2^{n+1} \pi n} = \frac{n\pi}{2^{n+1} n\pi \sqrt{n\pi}} =$$

$$= \frac{1}{2^n \cdot 2\sqrt{n\pi}} \approx \underline{\frac{1}{2^n \sqrt{n}}} \ldots$$

Dakle, $a_n \approx \frac{1}{2^n\sqrt{n}}$. Primjenimo kašijev kriterij:

$\lim\limits_{(n)} \sqrt[n]{a_n} = q \begin{matrix} <1 & Ⓚ \\ >1 & Ⓓ \end{matrix}$

$$\lim_{(n)} \sqrt[n]{\frac{1}{2^n\sqrt{n}}} = \lim_{(n)} \frac{1}{2\sqrt[n]{\sqrt{n}}} = \lim_{(n)} \frac{1}{2}\cdot\frac{1}{n^{\frac{1}{2n}}} = \frac{1}{2} < 1 \Rightarrow Ⓚ$$

Ⓩ Ispitati konvergenciju reda

$$\sum_{n=1}^{\infty} (\sqrt{2}-\sqrt[3]{2})(\sqrt{2}-\sqrt[5]{2})\cdots(\sqrt{2}-\sqrt[2n+1]{2})$$

$a_n = (\sqrt{2}-\sqrt[3]{2})(\sqrt{2}-\sqrt[5]{2})\cdots(\sqrt{2}-\sqrt[2n+1]{2})$

$a_{n+1} = a_n\cdot(\sqrt{2}-\sqrt[2n+3]{2})$

$\frac{a_{n+1}}{a_n} = (\sqrt{2}-\sqrt[2n+3]{2}) \Rightarrow \sqrt{2}\cdot\lim\limits_{(n)} 2^{\frac{1}{2n+3}} = \sqrt{2}-1 < 1$ pa Ⓡ Ⓚ.

Rabeov kriterij je:

$$\lim_{(n)} n\left(\frac{a_n}{a_{n+1}}-1\right) = n\cdot\left(\frac{1}{\sqrt{2}-\sqrt[2n+3]{2}}-1\right) = \lim_{(n)} \frac{n-\sqrt{2}-\sqrt[2n+3]{2}}{\sqrt{2}-\sqrt[2n+3]{2}} =$$

$$= \lim_{(n)} \frac{n-\sqrt{2}-1}{\sqrt{2}-1} = \lim_{(n)}\left(\frac{n}{\sqrt{2}-1}-1\right) = +\infty \begin{matrix} >1 & Ⓚ \\ <1 & Ⓓ \end{matrix}$$ pa red Ⓚ.

Ⓩ Ispitati konvergenciju reda: $\sum\limits_{n=1}^{\infty}\left(\frac{n-2}{n+2}\right)^{n(n+1)}$

Probajmo kašijev kriterij korijena: $\lim\limits_{(n)} \sqrt[n]{a_n} \begin{matrix} <1 & Ⓚ \\ >1 & Ⓓ \end{matrix}$

$$\lim_{(n)} \sqrt[n]{\left(\frac{n-2}{n+2}\right)^{n(n+1)}} = \lim_{(n)}\left(\frac{n-2}{n+2}\right)^{n+1} = \lim\left(1-\frac{4}{n+2}\right)^{n+1} =$$

$$= \lim_{(n)}\left(1-\frac{4}{n+2}\right)^{n+2-1} = \lim_{(n)}\left(1-\frac{4}{n+2}\right)\cdot\lim_{(n)}\left(\frac{1}{1-\frac{4}{n+2}}\right) = e^{-4}\cdot 1 = \frac{1}{e^4} < 1$$

pa red konvergira.

II način

$\lim\limits_{(n)}\left(\frac{n-2}{n+2}\right)^{n+1} = L \;/\; \circ\ln \Rightarrow \ln L = (n+1)\ln\left(\frac{n-2}{n+2}\right) \Rightarrow$

$\Rightarrow \ln L = \lim\limits_{(n)} (n+1)(\ln(n-2)-\ln(n+2))$ NE

$\Rightarrow \ln L = \lim\limits_{(n)} (n+1)\ln\left(\frac{n-2}{n+2}\right) \Rightarrow$ DA $\Rightarrow \left| \begin{matrix} \ln u \sim u-1,\ u\to 1 \\ \text{kod nas je } u = \frac{n-2}{n+2} \\ \lim\limits_{(n)} \frac{n-2}{n+2} = 1 \end{matrix} \right| \Rightarrow$

$\Rightarrow \ln L = \lim\limits_{(n)} (n+1)\left(\frac{n-2}{n+2}-1\right)$

$$\ln L = \lim_{(n)} (n+1)\frac{-4}{(n+2)} = \lim_{(n)} \frac{-4(n+1)}{(n+2)} = -4\cdot 1 = \ln L$$

$\ln L = -4 \Rightarrow L = e^{-4} = \frac{1}{e^4} < 1$ pa red konvergira.

ⓩ Ispitati konvergenciju reda: $\sum_1^{\infty} \frac{n^k}{a^n+b^n}$, $a>1, b>1$, $a>b>1$

Upotrijebit ćemo Košijev kriterij korijena:

$$\sqrt[n]{\frac{n^k}{a^n+b^n}} = \frac{\sqrt[n]{n^k}}{\sqrt[n]{a^n+b^n}} = \frac{\sqrt[n]{n^k}}{a\sqrt[n]{1+\left(\frac{b}{a}\right)^n}}, \quad \left(\frac{b}{a}\right)<1 \text{ pa } \left(\frac{b}{a}\right)^n \to 0,\ n\to\infty$$

$$\lim_{(n)} \sqrt[n]{a_n} = \frac{\sqrt[n]{n^k}}{a\sqrt[n]{1+\underbrace{\left(\frac{b}{a}\right)^n}_{\to 0}}}\ ; \quad \lim_{(n)} \sqrt[n]{n^k} = \lim_{(n)} \frac{(n+1)^k}{n^k} = \lim_{(n)} \left(1+\frac{1}{n}\right)^k = 1$$

$a_n = n^k$

$\lim_{(n)} \frac{1}{a} < 1$ jer je $a>1$ pa red konvergira.

ⓩ Ispitati konvergenciju reda: $\sum_1^{\infty} \frac{a+(-1)^n b}{a^n}$, $a>b>1$.

Uzmimo parne n i upotrijebimo Košijev kriterij!

$$a_{2n} = \frac{a+b}{a^{2n}}$$

$$\Delta(\text{koši}) = \lim_{(n)} \sqrt[2n]{a_{2n}} = \frac{\sqrt[2n]{a+b}}{\sqrt[2n]{a^{2n}}} = \lim_{(n)} \frac{\sqrt[2n]{a+b}}{a} =$$

$$= \frac{1}{a} < 1 \Rightarrow (k)$$

Neparni: a_{2n-1}

$$\lim_{(n)} \sqrt[2n-1]{a_{2n-1}} = \frac{\sqrt[2n-1]{a-b}}{\sqrt[2n-1]{a^{2n-1}}} = \lim_{(n)} \frac{\sqrt[2n-1]{a-b}}{a} = \frac{1}{a} < 1 \Rightarrow (k)$$

ⓩ Ispitati konvergenciju reda $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{(a+\sqrt{1})(a+\sqrt{2})\cdot\ldots\cdot(a+\sqrt{n})}$

D'Alomberov kriterij!

$$a_{n+1} = \frac{\sqrt{n+1}}{(a+\sqrt{1})(a+\sqrt{2})\cdot\ldots\cdot(a+\sqrt{n})(a+\sqrt{n+1})}$$

$$\frac{a_{n+1}}{a_n} = \frac{\sqrt{n}\cdot\sqrt{n+1}}{(\sqrt{n+1}+a)} = \frac{\sqrt{n}\sqrt{n+1}}{\sqrt{n+1}\left(1+\frac{a}{\sqrt{n+1}}\right)} = \frac{\sqrt{n}}{1+\frac{a}{\sqrt{n+1}}} = \frac{\sqrt{n}}{\frac{\sqrt{n+1}+a}{\sqrt{n+1}}} =$$

$\frac{\sqrt{n+1}}{\sqrt{n}} \cdot \frac{1}{a+\sqrt{n+1}} = \left(\sqrt{1+1/n}\right) \cdot \frac{1}{a+\sqrt{n+1}} = \frac{\sqrt{1}}{\infty} = \frac{1}{\infty} = 0 < 1$ Ⓚ.

Ⓩ $\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{2n!!} \right)^p$

$a_{n+1} = \left(\frac{(2n-1)!! \cdot (2n+1)}{2n!! \, (2n+2)} \right)^p , \quad \frac{a_{n+1}}{a_n} = \left(\frac{2n+1}{2n+2} \right)^p \Big/ \odot \lim_{(n)} \Rightarrow$

$\Rightarrow \lim_{(n)} \frac{a_{n+1}}{a_n} = \lim_{(n)} \left(\frac{2+\frac{1}{n}}{2+\frac{2}{n}} \right)^p = \underline{1}.$

Tako da Δ(koši) NE DOLAZI u obzir, probajmo RABEA:

$\lim_{(n)} n \left(\frac{a_n}{a_{n+1}} - 1 \right) = \lim_{(n)} n \left(\left(\frac{2n+2}{2n+1} \right)^p - 1 \right) = \left| \left(\frac{2n+2}{2n+1} \right)^p = e^{\ln\left(\frac{2n+2}{2n+1}\right)^p} \right| =$

$= \lim_{(n)} n \left(e^{\ln\left(1+\frac{1}{2n+1}\right)^p} - 1 \right) = \left| \underset{n\to\infty}{e^u - 1 \sim u}, u \to 0 \right| = \lim_{(n)} n \ln\left(1+\frac{1}{2n+1}\right)^p \to +\infty.$

$\lim_{(n)} np \underbrace{\ln\left(1+\underset{\to 0}{\frac{1}{2n+1}}\right)} = \lim_{(n)} np \frac{1}{2n+1} = \frac{p}{2}$

$\underset{u\to 0}{\ln(1+u) \sim u}$

Za $\frac{p}{2} > 1 \Rightarrow$ Ⓚ

$\frac{p}{2} < 1 \Rightarrow$ Ⓓ

$\frac{p}{2} = 1 \Rightarrow p=2$ – Za $p=2$ moramo ispitati red iz početka.

Imamo: $\sum_{n=1}^{\infty} \left(\frac{(2n-1)!!}{2n!!} \right)^2 \Rightarrow$ ŠTIRLING $\Rightarrow$

$a_n = \left(\frac{1\cdot 3 \cdot \ldots \cdot 2n-1}{2\cdot 4 \cdot \ldots \cdot 2n} \right)^2 = \left(\frac{1\cdot 2\cdot 3\cdot 4 \cdot \ldots \cdot 2n-1 \cdot 2n}{2^n \cdot n! \cdot 2^n \cdot n!} \right)^2 = \left(\frac{2n!}{4^n (n!)^2} \right)^2 \Rightarrow$

$a_n = \left(\frac{2n!}{4^n (n!)} \right)^2 = |\check{S}.| = \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{4\pi n}}{4^n \left(\frac{2n}{e}\right)^{2n} 2\pi n} = \frac{\left(\frac{2n}{e}\right)^{2n} \sqrt{2\pi n}}{\left(\frac{2n}{e}\right)^{2n} \cdot 2\pi n} = \frac{\sqrt{\pi n}}{\pi^n} =$

$= \frac{1}{\sqrt{\pi n}} \sim \frac{1}{n^{1/2}}$

$\Delta(\log) = \lim_{(n)} \frac{\frac{1}{2} \ln n}{\ln n} = \frac{1}{2} < 1$ pa red Ⓓ za $\underline{p=2}$.

(Z) Ispitati konvergenciju reda $\sum_{n=1}^{\infty} 5^{\sqrt[3]{n}}$!

$a_n = 5^{n^{\frac{1}{3}}}$ – uporediti ovo sa $b_n = \frac{1}{n^2}$

$a_n > 0$ $\qquad$ $b_n > a_n > 0$

$$5^{n^{\frac{1}{3}}} < \frac{1}{n^2}$$

$$n^2 < 5^{n^{\frac{1}{3}}} \Rightarrow |n \equiv x^3|$$

$$x^6 < 5^x$$

$$x < 5^{\frac{x}{6}}$$ za $x \gg$ (T) po red $\sum_1^{\infty} 5^{n^{\frac{1}{3}}}$ (K).

(Z) Ispitati konvergenciju reda:

$$\sum_{n=1}^{\infty} \left(e^{\frac{1}{n+1}} - 1\right)^p, \ p \in R.$$

$e^u - 1 \sim u$, $u \to 0$ $\qquad a_n \approx \left(\frac{1}{n+1}\right)^p$

D'Alamber : $\frac{a_{n+1}}{a_n} = \frac{\left(\frac{1}{n+2}\right)^p}{\left(\frac{1}{n+1}\right)^p} = \left(\frac{\frac{1}{n+2}}{\frac{1}{n+1}}\right)^p = \left(\frac{n+1}{n+2}\right)^p = \left(1 + \frac{-1}{n+2}\right)^p$

$\lim_{(n)} \left(1 + \left(-\frac{1}{n+2}\right)\right)^p = 1^p = 1 \Rightarrow$ D'Alamberov kriterij ne daje rješenje za ovaj slučaj.

$a_n = \left(\frac{1}{n+1}\right)^p \cong \frac{1}{n^p} \Rightarrow \Delta(\log) = \frac{p \ln n}{\ln n} = p > 1$ (K)

$p \leq 1$ (D)

(Z) Ispitati konvergenciju reda:

$$\sum_1^{\infty} \frac{1}{n^{\alpha}} \ln\left(n \sin\frac{1}{n}\right),$$

$a_n = \frac{1}{n^{\alpha}} \ln\left(n \sin\frac{1}{n}\right) = \frac{1}{n^{\alpha}} \ln\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)$

$a_n = \frac{1}{n^{\alpha}}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}} - 1\right) = \frac{1}{n^{\alpha}}\left(\frac{\sin\frac{1}{n} - \frac{1}{n}}{\frac{1}{n}}\right)$

| $\ln u \sim u - 1$, $u \to 1$, a znamo da $\frac{\sin x}{x} \to 1$, $x \to \infty$ |

Kako je $\lim_{x \to 0} \frac{x - \sin x}{x^3} = \frac{1}{6}$ to je $x - \sin x \sim \frac{x^3}{6}$ tj., $\sin x - x \approx -\frac{x^3}{6}$, $x \to 0$

pa imamo da je: $a_n = \frac{1}{n^{\alpha}}\left(\frac{-\frac{6}{n^3}}{\frac{1}{n}}\right) = \frac{1}{n^{\alpha}} \cdot \frac{-6n}{n^3} =$

$$= \frac{1}{n^\alpha}\left(-\frac{6}{n^2}\right) = \frac{-6}{n^{\alpha+2}} \sim \frac{-1}{n^{\alpha+2}}$$

$$\Delta(\log) = \frac{(\alpha+2)\ln n}{\ln n} = \alpha+2 \begin{array}{l} >1 \Rightarrow (K) \\ \leq 1 \Rightarrow (D) \end{array} \Big\} \text{ tj., za } \alpha>-1 \Rightarrow (K), \text{ za } \alpha\leq -1 \Rightarrow (D)$$

(Z) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} + \ln\left(1+\frac{1}{n}\right)$, $a_n = \frac{1}{\sqrt{n}} + \ln\left(1+\frac{1}{n}\right) = \left|\ln n \sim n-1, n\to 1\right| =$

$$= \frac{1}{\sqrt{n}} + \left(\left(1+\frac{1}{n}\right)-1\right) = \frac{1}{\sqrt{n}} + \frac{1}{n} = \frac{n+\sqrt{n}}{n\sqrt{n}} = \frac{n\left(1+\frac{\sqrt{n}}{n}\right)}{n\sqrt{n}} = \frac{1+\frac{1}{\sqrt{n}}}{\sqrt{n}}$$

$$\Delta(\log) = \frac{\ln\left(\frac{n\sqrt{n}}{n+\sqrt{n}}\right)}{\ln n} = \frac{\ln n\sqrt{n} - \ln(n+\sqrt{n})}{\ln n} = \frac{\ln\left(\frac{\sqrt{n}}{1+\frac{1}{\sqrt{n}}}\right)}{\ln n} =$$

$$= \frac{\frac{1}{2}\ln n - \overbrace{\ln\left(1+\frac{1}{\sqrt{n}}\right)}^{\to 0}}{\ln n} = \frac{1}{2} < 1 \text{ pa red } (D) \text{ ili}$$

$$a_n = \frac{1}{\sqrt{n}} + \ln\left(1+\frac{1}{n}\right)$$

$\ln(1+x) \sim x - \frac{x^2}{2}$ kad $x\to 0$ imamo: $a_n \sim \frac{1}{\sqrt{n}} - \frac{1}{2n^2} =$

$$= \frac{2n\sqrt{n}+2n-1}{2n^2} = \frac{2n(\sqrt{n}+1)-1}{2n^2} = \underbrace{\frac{\sqrt{n}+1}{n} - \frac{1}{2n^2} > \frac{1}{n}}_{(1)} \to 0$$

(1) $\Rightarrow \frac{1}{\sqrt{n}} + \frac{1}{n} - \frac{1}{2n^2} > \frac{1}{n}$, $\frac{1}{\sqrt{n}} > \frac{1}{2n^2}$, $2n^2 > \sqrt{n} \Rightarrow 4n^4 > n^2$ (T)

$\Rightarrow$ <u>divergira</u>.

(Z) Ispitati konvergenciju reda $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \ln\left(1+\frac{1}{n}\right)$!

$\ln\left(1+\frac{1}{n}\right) \sim \frac{1}{n}$, $\frac{1}{n}\to 0$

$\ln\left(1+\frac{1}{n}\right) \sim \frac{1}{n} - \frac{1}{2n^2}$

$a_n = \frac{1}{\sqrt{n}} - \frac{1}{n} + \frac{1}{2n^2}$. Pretpostavimo da ovaj red (K)! Tada je

$$\frac{1}{\sqrt{n}} - \frac{1}{n} + \frac{1}{2n^2} < \frac{1}{2n^2}$$

$\frac{1}{\sqrt{n}} - \frac{1}{n} < 0 \Rightarrow \frac{1}{\sqrt{n}} < \frac{1}{n} \Big/ \cdot \frac{\sqrt{n}}{\sqrt{n}} \Rightarrow \sqrt{n} > n$ (⊥) NE (K)

Pretpostavimo da red divergira:

$$\underbrace{\underbrace{\frac{1}{\sqrt{n}}-\frac{1}{n}+\frac{1}{2n^2}}_{a_n} > \underbrace{-\frac{1}{n}}_{b_n}}$$

proizvoljno ih označim.

$$\frac{1}{\sqrt{n}} > -\frac{1}{2n^2}$$

$$\sqrt{n} < -2n^2 \quad /^2$$

$\underline{n < 4n^4}$. (T) $\Rightarrow$ red divergira!

(Z) Odrediti sumu reda:

(1) $\sum_{n=1}^{\infty}\left(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n}\right)$.

n-tu parcijalnu sumu reda označimo sa S_n. Dakle,

$$S_n=\sum_{k=1}^{n}\left(\sqrt{k+2}-2\sqrt{k+1}+\sqrt{k}\right)=\sum_{k=1}^{n}\sqrt{k+2}-\sum_{k=1}^{n}2\cdot\sqrt{k+1}+\sum_{k=1}^{n}\sqrt{k}=$$

$$=\left|\begin{matrix}\text{vršimo reduciranje}\\ \text{suma na } \sqrt{k}\end{matrix}\right|=\sum_{k=3}^{n+2}\sqrt{k}-2\sum_{k=2}^{n+1}\sqrt{k}+\sum_{k=1}^{n}\sqrt{k}=$$

$$=\sqrt{k+2}+\sqrt{k+1}+\sum_{k=3}^{n}\sqrt{k}-2\sqrt{k+1}-2\sqrt{2}-2\sum_{k=3}^{n}\sqrt{k}+1+\sqrt{2}+\sum_{k=3}^{n}\sqrt{k}=$$

$=\sqrt{k+2}-\sqrt{k+1}-\sqrt{2}+1$. Dakle, $S_n=1+\sqrt{2+k}-\sqrt{k+1}-\sqrt{2}$, pa je suma reda (1) tada: $S=\lim_{(n)}\left(1-\sqrt{2}+\sqrt{k+2}-\sqrt{k+1}\right)=$

$$=\lim_{(n)}(1-\sqrt{2})+\lim_{(n)}\left(\sqrt{k+2}-\sqrt{k+1}\right)\Big/\cdot\frac{\sqrt{k+2}+\sqrt{k+1}}{\sqrt{k+2}+\sqrt{k+1}}=$$

$$=1-\sqrt{2}+\lim_{(n)}\frac{k+2-k-1}{\sqrt{k+2}+\sqrt{k+1}}=1-\sqrt{2}+\lim_{(n)}\frac{1}{\sqrt{k+2}+\sqrt{k+1}}=\underline{1-\sqrt{2}}.$$

($\frac{1}{\sqrt{k+2}+\sqrt{k+1}} \to 0$)

Suma reda (1) je $\underline{S=1-\sqrt{2}}$.

(Z) Ispitati konvergenciju i naći sumu reda:

$$\frac{1}{\sqrt{1\cdot3}}+\frac{1}{\sqrt{3\cdot5}}+\frac{1}{\sqrt{5\cdot7}}+\ldots \text{ tj. } \sum_{1}^{\infty}\frac{1}{\sqrt{(2n-1)(2n+1)}}.$$

$$S_n=\sum_{k=1}^{n}\frac{1}{\sqrt{(2n-1)(2n+1)}}=\sum_{k=1}^{n}\frac{1}{\sqrt{2k-1}}\cdot\sum_{k=1}^{n}\frac{1}{\sqrt{2k+1}}=$$

$$= \sum_{k=1}^{n} \frac{1}{\sqrt{2k-1}} \cdot \left(\sum_{k=1}^{n-2} \frac{1}{\sqrt{2k-1}} + \frac{1}{\sqrt{2k}} + \frac{1}{\sqrt{2k+1}} \right)$$

međutim ovo ne pomaže jer se članovi međusobno ne mogu kratiti!

$\sum_{1}^{\infty} \frac{1}{\sqrt{(2n-1)(2n+1)}} = \sum_{1}^{\infty} \frac{1}{\sqrt{4n^2-1}} > 0$, $a_n = \frac{1}{\sqrt{4n^2-1}} > 0$ - red je pozitivan pa je za dovoljno velike n, $a_n \sim \frac{1}{2n}$ jer je:

$$\frac{1}{\sqrt{4n^2-1}} = \frac{1}{2n\underbrace{\sqrt{1-\frac{1}{4n^2}}}_{\text{mali broj}}}$$

kako je $\frac{1}{2n} \approx a_n$ to je red divergentan i suma mu je: $+\infty$.

② Ispitati konvergenciju reda datog sa:

$$\frac{\cos x^n}{1^2} + \frac{\cos x^n}{2^2} + \dots + \frac{\cos x^n}{n^2}, x \in \mathbb{R} \quad \text{ili}$$

$$\cos x^n \cdot \sum_{n=1}^{\infty} \frac{1}{n^2}. \quad (1)$$

RJ:

$|\cos x^n| < 1$, a $\sum_{1}^{\infty} \frac{1}{n^2}$ je konvergentan pa je:

$a_n = \frac{|\cos x^n|}{n^2} < \frac{1}{n^2}$ → ovo je uporedni stav prema kojem je red (1) konvergentan.

② Ispitati konvergenciju reda (*) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha}$, $\alpha \in \mathbb{R}$!

Alternativni redovi su oni kojima članovi mijenjaju predznak. Za ovakve redove koristi se Lajbnicov kriterij, koji kaže: „Naizmjenični red konvergira ako je: $b_n \geq b_{n+1}$, $n = 1, 2, 3, \dots$ i $\lim_{(n)} b_n = 0$!"

$$(*) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^\alpha}, \alpha \in \mathbb{R}.$$

Ispitajmo uslovnu konvergenciju reda! Za $\alpha \leq 0$ red divergira jer $a_n \not\to 0$, $n \to +\infty$.

Za $\alpha > 0 \Rightarrow$ (1) $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n^{\alpha}} = \frac{-1}{1^{\alpha}} + \frac{1}{2^{\alpha}} - \frac{1}{3^{\alpha}} + \frac{1}{4^{\alpha}} - \frac{1}{5^{\alpha}} + \dots + \frac{1}{(2n)^{\alpha}}$

Za $\alpha < 0 \Rightarrow$ (1) $= -1^{\alpha} + 2^{\alpha} - 3^{\alpha} + 4^{\alpha} - 5^{\alpha} + \dots + (2n)^{\alpha}$

Za $\alpha < 0 \Rightarrow a_n = n^{\alpha}(-1)^{n-1} \not\to 0$, $n \to \infty$

Primjenimo sada Lajbnicov kriterij:

$a_n = \frac{1}{n^{\alpha}}$, $a_{n+1} = \frac{1}{(n+1)^{\alpha}}$

1° $a_n > a_{n+1}$ (T)

2° $\lim_{(n)} a_n = 0$, za $\alpha > 0$ } (K)

3° $\lim_{(n)} a_n \neq 0$, za $\alpha < 0$ } (D)

→ Uslovna konvergencija

Red (*) apsolutno konvergira, ako konvergira red $\sum_{1}^{\infty} \frac{1}{n^{\alpha}}$

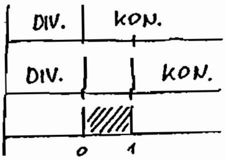
Ispitujemo nekim od kriterija.

Red $\sum_{1}^{\infty} \frac{1}{n^{\alpha}}$ konvergira za $\alpha > 1$, a divergira za $\alpha \leq 1$, prema uporednom kriteriju.

Relativna (K)

Apsolutna (K)

Semi (K)



(2) Ispitati konvergenciju reda: $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{\ln n}$!

Red $\sum_{1}^{\infty} a_n(-1)^n$ konvergira ako je:

1° $\lim_{(n)} a_n \to 0$

2° $a_n > a_{n+1}$, $n = 1, 2, 3, \dots$

Imamo

1° $\lim_{(n)} \frac{1}{\ln n} \to 0$ (T) , $a_n = \frac{1}{\ln n}$, $a_{n+1} = \frac{1}{\ln(n+1)}$

2° $a_n > a_{n+1}$ (T) pa red konvergira!

Ovo je bila provjera za uslovnu konvergenciju!

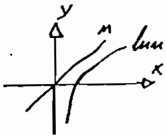
Apsolutna konvergencija:

$\sum_{1}^{\infty} \frac{1}{\ln n}$, $a_n = \frac{1}{\ln n} > 0$ poz. red.

$\Delta(\log) = \frac{\ln \ln n}{\ln n} = \left| \ln n = t,\ t \to \infty \right| = \frac{\ln t}{t} = \left| \begin{matrix} \ln t = x, \\ e^x = t \end{matrix} \right| = \frac{x}{e^x}$.

$\lim_{x \to \infty} \frac{x}{e^x} = 0 < 1 \Rightarrow$ da red (D)!

Možemo upotrijebiti i uporedni kriterij:



$\frac{1}{\ln n} > \frac{1}{n} \Rightarrow n > \ln n$ (T) za $n \gg$ pa red (D)

(Z) Ispitati konvergenciju reda: $\sum_{n=2}^{\infty} \frac{1}{\ln^2 n} \cos \frac{\pi n^2}{n+1}$.

$a_n = \frac{1}{(\ln n)^2} \cos \frac{\pi n^2}{n+1}$. kako je $\left| \cos \frac{\pi n^2}{n+1} \right| < 1$ to je:

$\frac{1}{(\ln n)^2} \cos \frac{\pi n^2}{n+1} < \frac{1}{(\ln n)^2}$ | međutim ovo je nepotrebno.

Trebamo srediti $\cos \frac{\pi n^2}{n+1}$:

$\cos \frac{\pi n^2}{n+1} = \cos \frac{\pi (n^2 - 1 + 1)}{n+1} = \cos \frac{\pi [(n+1)(n-1) + 1]}{n+1} = \cos \left[(n-1)\pi + \frac{\pi}{n+1} \right] =$

$= \underbrace{\cos (n-1)\pi}_{(-1)^{n-1}} \cos \frac{\pi}{n+1} - \underbrace{\sin (n-1)\pi}_{= 0} \sin \frac{\pi}{n+1}$

$\cos (n-1)\pi \cdot \cos \frac{\pi}{n+1} = (-1)^{n-1} \cdot \cos \frac{\pi}{n+1}$, kako je $\cos \frac{\pi}{n+1} \sim 1$ jer $\frac{\pi}{n+1} \to 0$ imamo da je: $a_n = \frac{1}{\ln^2 n} (-1)^{n-1}$. Sada smo dobili red: $\sum_{2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^2}$

$\Rightarrow$ Uslovna konvergencija (Lajbnic):

1° $a_n > a_{n+1} \Rightarrow \frac{1}{(\ln n)^2} > \frac{1}{\ln (n+1)^2}$ (T)

2° $\lim_{(n)} \frac{1}{(\ln n)^2} = 0$ (T)

} (K)

Apsolutna konvergencija:

$$\sum_{n=2}^{\infty}\frac{1}{\ln^2 n} \Rightarrow \frac{a_{n+1}}{a_n}=\frac{\frac{1}{\ln^2(n+1)}}{\frac{1}{\ln^2 n}}=\ln^2\left(\frac{n}{n+1}\right) \text{ - ne može!}$$

$$\Delta(\log)=\frac{\ln\frac{1}{a_n}}{\ln n}=\frac{\ln\ln^2 n}{\ln n}=\left|\ln n=t\right|=\frac{\ln t}{t}=\left|\begin{matrix}\ln t=x\\ e^x=t\end{matrix}\right|=\frac{x}{e^x};$$

$$\lim_{(x)}\frac{x}{e^x}=0<1 \Rightarrow \text{(D)}.$$

(2) Ispitati konvergenciju reda: $\sum_{n=1}^{\infty}(-1)^n\left(\frac{(2n-1)!!}{2n!!}\right)^{\alpha}$, $\alpha\in\mathbb{R}$!

Koristimo Štirlingovu formulu: $n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$

$$\frac{(2n-1)!!}{2n!!}=\frac{1\cdot3\cdot5\cdot\ldots\cdot 2n-1}{2\cdot4\cdot6\cdot\ldots\cdot2n}=\frac{1\cdot2\cdot3\cdot\ldots\cdot(2n-1)\cdot2n}{2^n n!\cdot2^n n!}=\frac{2n!}{2^{2n}(n!)^2}=\frac{\left(\frac{2n}{e}\right)^{2n}\sqrt{4\pi n}}{2^{2n}\cdot\left(\frac{n}{e}\right)^{2n}2\pi n}=$$

$$=\frac{\sqrt{4\pi n}}{2\pi n}=\frac{2}{\sqrt{4\pi n}}=\frac{1}{\sqrt{\pi n}}\sim\frac{1}{n^{\frac{1}{2}}} \Rightarrow a_n\approx\left(\frac{1}{n^{\frac{1}{2}}}\right)^{\alpha}=\frac{1}{n^{\frac{\alpha}{2}}}..$$

$$\sum_{n=1}^{\infty}(-1)^n\left(\frac{(2n-1)!!}{2n!!}\right)^{\alpha}=\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^{\alpha/2}}$$

* Apsolutna konvergencija

$\sum_{n=1}^{\infty}\frac{1}{n^{\alpha/2}}$ za $\alpha>2$ red (K)

za $\alpha\le2$ red (D)

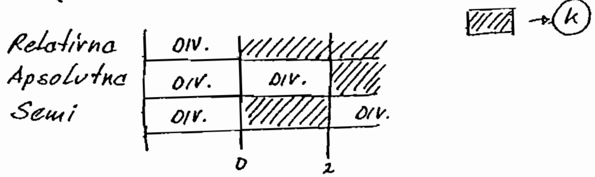
* Uslovna (relativna): Lajbnic:

$\left(\frac{(2n-1)!!}{2n!!}\right)^{\alpha}\to0$ za $\alpha>0$, za $\alpha\le0$ red divergira

Zato ispitujemo za $\alpha>0$:

$$\frac{a_n}{a_{n+1}}=\frac{2n+2}{2n+1}=|Z=\frac{\frac{(2n-1)!!}{2n!!}}{\frac{(2n-1)!!(2n+1)}{(2n)!!(2n+2)}}=\frac{2n+1+1}{2n+1}>1\Rightarrow a_n>a_{n+1}..$$

Red konvergira za $\alpha>0$!



(Z) Ispitati konvergenciju reda: $\sum_{1}^{\infty} \frac{(-1)^n}{(\sqrt{n}+(-1)^{n-1})^p}$, $p \in R$.

$a_n = \frac{(-1)^n}{(\sqrt{n}+(-1)^{n-1})^p}$, $p \in R$

Za $p<0$ red će divergirati jer mu tada opšti član teži nuli!

Napišimo a_n u obliku: $a_n = \frac{(-1)^n}{(\sqrt{n})^p\left(1+\frac{(-1)^{n-1}}{\sqrt{n}}\right)^p} =$

$= \frac{(-1)^n}{(\sqrt{n})^p}\left(1+\frac{(-1)^{n-1}}{\sqrt{n}}\right)^{-p}$ Razvijmo sada f-ju $(1+x)^\alpha$ u

Maklorenov red! (*) $(1+x)^\alpha \sim 1+\alpha x+\frac{\alpha(\alpha-1)}{2!}x^2$, kod $x \to 0$ do 3 člana.

Sada je: $\left(1+\frac{(-1)^{n-1}}{\sqrt{n}}\right)^{-p} \sim 1-p\frac{(-1)^{n-1}}{\sqrt{n}}+\frac{p(p+1)}{2}\cdot\frac{1}{n}$.

Izlazi da je: $a_n \sim \frac{(-1)^n}{n^{p/2}}\left(1-\frac{p(-1)^{n-1}}{n^{1/2}}+\frac{p(p+1)}{2}\cdot\frac{1}{n}\right) \Rightarrow$

$$a_n \sim \underbrace{\frac{(-1)^n}{n^{p/2}}}_{1^\circ} - \underbrace{p\frac{1}{n^{\frac{p+1}{2}}}}_{2^\circ} + \underbrace{\frac{p(p+1)}{2}\cdot\frac{(-1)^{n-1}}{n^{\frac{p+2}{2}}}}_{3^\circ}$$

Za sve $1^\circ, 2^\circ$ i 3° moramo ispitati uslovnu i apsolutnu konvergenciju!

1° $\frac{(-1)^{n-1}}{n^{p/2}} \Rightarrow a_n = \frac{1}{n^{p/2}}$ za $p>2$ (K), za $p \le 2$ (D) } Aps. (K)

a) $\lim_{(n)} \frac{1}{n^{p/2}} \to 0$ za $p>0$ (T)

b) $\frac{a_n}{a_{n+1}} = \left(\frac{n+1}{n}\right)^{p/2} > 1$ za $p>0$ (T)

} (K) Uslovno

2° $\frac{p}{n^{\frac{p+1}{2}}} \Rightarrow a_n = \frac{p}{n^{\frac{p+1}{2}}}$ za $p>1$ (K), za $p \le 1$ (D) } Aps. (K)

a) $\frac{a_n}{a_{n+1}} = \left(1+\frac{1}{n}\right)^{p+1} > 1$ za $p>1$ (T)

b) $\lim_{(n)} \frac{p}{n^{\frac{p+1}{2}}} \to 0$ za $p>-1$ (T)

} Relativno (K)

3° $\frac{p(p+1)}{2}\cdot\frac{(-1)^{n-1}}{n^{\frac{p+2}{2}}} \Rightarrow a_n \approx \frac{1}{n^{\frac{p+2}{2}}}$ za $p>0$ (K), za $p\le 0$ (D) } Aps.(K)

a) $\frac{a_n}{a_{n+1}} = \left(\frac{n+1}{n}\right)^{\frac{p+2}{2}} > 1$ za $p>-2$ (T)

b) $\lim\limits_{(n)} \frac{1}{n^{\frac{p+2}{2}}} \to 0$ za $p>-2$ (T) } Rel.(K)

$a_n \sim$	1° −	2° +	3°	ZAKLJUČAK
Apsolutna	$p>2$	$p>1$	$p>0$	$p>2$ (K)
Uslovna	$p>0$	$p>-1$	$p>-2$	$p>0$ (K)
Semi	$0<p<2$			

$0<p<2$: prostor gdje je red apsolutno divergentan i uslovno konvergentan!

(Z) Ispitati konvergenciju reda: $\sum\limits_1^{\infty} \ln\left(1+\frac{\sqrt[n]{n^2+1}\,(-1)^n}{n^\alpha}\right)$!

$$a_n = \ln\left(1+\frac{\sqrt[n]{n^2+1}\,(-1)^n}{n^\alpha}\right)$$

1° za $\alpha<0$, $a_n \to +\infty$ pa red divergira.

2° $a_n = \ln\left(1+\frac{\sqrt[n]{n^2+1}\,(-1)^n}{n^\alpha}\right)$. kako $\sqrt[n]{n^2+1} \to 1$ kad $n\to+\infty$ jer je $(n^2+1)^{\frac{1}{n}} = n^{\frac{2}{n}}\left(1+\frac{1}{n^2}\right)$ pa je $\lim\limits_{(n)} n^{\frac{2}{n}}\left(1+\frac{1}{n^2}\right) = 1$, pa $\frac{\sqrt[n]{n^2}+(-1)^n}{n^\alpha} \to 0$

$$a_n = \ln\left(1+\frac{\sqrt[n]{n^2+1}}{n^\alpha}(-1)^n\right) = \left|\begin{array}{l}\ln(1+x)\sim x,\\ x\to 0 \quad \sim x-\frac{x^2}{2}\end{array}\right| \sim \frac{\sqrt[n]{n^2+1}}{n^\alpha}(-1)^n - \frac{\sqrt[n]{(n^2+1)^2}}{2n^{2\alpha}}$$

pa je zbog $\sqrt[n]{n^2+1} \to 1$, $n\to+\infty$; $a_n \sim \frac{(-1)^n}{n^\alpha} - \frac{1}{2n^{2\alpha}}$ ...

1° $a_n = \frac{1}{n^\alpha}$ za $\alpha>1$ (K), za $\alpha<1$ (D) } (K)

a) $\frac{a_n}{a_{n+1}} = \left(\frac{n+1}{n}\right)^\alpha > 1$ (T)

b) $\lim\limits_{(n)} \frac{1}{n^\alpha} \to 0$ (T) } Usl.(K)

za $\alpha>0$

2° $a_n = \frac{1}{n^{2\alpha}} \Rightarrow \frac{1}{n} \to$ (D), $\frac{1}{n^2} \to$ (K) tj., $2\alpha>1 \Rightarrow \alpha>\frac{1}{2}$ (K)

a) $\frac{a_n}{a_{n+1}} \Rightarrow 2\alpha>0 \Rightarrow \alpha>0$ (K)!

$$a_n \approx \frac{(-1)^n}{n^\alpha} - \frac{1}{2n^{2\alpha}}$$

			zaključak
Uslovna	$\alpha > 0$	$\alpha > 0$	$\alpha > 0$
Aps. (k)	$\alpha > 1$	$\alpha > \frac{1}{2}$	$\alpha > 1$
Semi (k)			$0 < \alpha < 1$

(Z) Ispitati konvergenciju reda: $\sum_{n=1}^{\infty}\left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)^\alpha$, $\alpha \in \mathbb{R}$.

$\ln(1+\frac{1}{n}) \equiv \ln(1+x) \sim x$, $x \to 0$ ili $x - \frac{x^2}{2}$, $x \to 0$

$$a_n = \left(\frac{1}{n} - \ln\left(1+\frac{1}{n}\right)\right)^\alpha = \left(\frac{1}{n} - \left(\frac{1}{n} - \frac{1}{2n^2}\right)\right)^\alpha = \frac{1}{2n^{2\alpha}}$$

$a_n = \frac{1}{2n^{2\alpha}} \Rightarrow$ (k) za $2\alpha > 1$, $\alpha > \frac{1}{2}$

$\Rightarrow$ (D) za $\alpha \le \frac{1}{2}$

(Z) Ispitati konvergenciju reda: (*) $\sqrt{2} + \sum_{1}^{\infty} \sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}}$.

$\alpha_n = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}$ Očito je $\alpha_n > 0$. S druge strane je:

$\alpha_1 = \sqrt{2} < 2$

$\alpha_2 = \sqrt{2+\sqrt{2}} < \sqrt{2+2} < 2$

$\vdots$

$\alpha_n = \sqrt{2+\alpha_{n-1}} < 2$

Odavde možemo pretpostaviti da je $\alpha_n < 2$ pa je u tom slučaju $0 < \alpha_n < 2$

$$\alpha_{n+1} = \sqrt{2+\alpha_n} \Rightarrow \alpha_{n+1} - \alpha_n = \sqrt{2+\alpha_n} - \alpha_n \overset{\text{RACIONALIZACIJA}}{=} \frac{(\alpha_n+1)(2-\alpha_n)}{\sqrt{2+\alpha_n}+\alpha_n} > 0 \Rightarrow$$

$\Rightarrow \alpha_{n+1} > \alpha_n$ pa dati niz α_n konvergira.

$/2$ $\alpha_{n+1} = \sqrt{\alpha_n + 2} \Rightarrow \alpha = \sqrt{\alpha+2} \Rightarrow \boxed{\alpha = 2}$

Dakle, $\lim \alpha_n = 2$. Odavde imamo da je:

(*) $\sqrt{2-\alpha_n} = \sqrt{2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}} > 0$ pa je red pozitivan.

S druge strane je: $\sqrt{2} = 2\cos\frac{\pi}{4}$ pa je: $\sqrt{2-\sqrt{2}} = \sqrt{2-2\cos\frac{\pi}{4}} =$

$$= \sqrt{2(1-\cos\frac{\pi}{4})} = \sqrt{2\cdot 2\sin^2\frac{\pi}{8}} = \sqrt{4\cdot \sin^2\frac{\pi}{2^3}} = 2\sin\frac{\pi}{2^3}.$$

Dalje je: $\sqrt{2-\sqrt{2+\sqrt{2}}} = \sqrt{2-\sqrt{2+2\cos\pi/4}} = \sqrt{2-\sqrt{2(1+\cos\frac{\pi}{4})}} = \sqrt{2-\sqrt{2\cdot 2\cos^2\frac{\pi}{8}}} =$

$= \sqrt{2-2\cos^2\frac{\pi}{8}} = 2\sin^2\frac{\pi}{16} = 2\sin\frac{\pi}{2^4}$

Možemo zaključiti da je $a_n = 2\sin\frac{\pi}{2^{n+1}}$. Dakle, red je $\sqrt{2} + \sum_1^\infty 2\sin\frac{\pi}{2^{n+1}}$! Sada je $a_n = 2\sin\frac{\pi}{2^{n+1}} \sim \left|\begin{matrix}\sin x \sim x\\ x\to 0\end{matrix}\right| \sim$

$\sim 2\frac{\pi}{2^{n+1}} = \frac{\pi}{2^n}$.

$a_n = \frac{\pi}{2^n}$ - pošto imam stepen idem preko košija!

$\Delta(koš) = \sqrt[n]{a_n} = \sqrt[n]{\frac{\pi}{2^n}} = \frac{1}{2} < 1 \Rightarrow$ dati red konvergira.

ⓩ Ispitati apsolutnu konvergenciju reda:

$$(*) \quad \sum_{n=1}^\infty (-1)^{n-1}\left[\sqrt{2-\left(\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}\right)^\alpha}\right]^\beta; \quad \alpha,\beta \in \mathbb{R}.$$

koristeći rezultate iz prethodnog primjera znamo da niz $\alpha_n = \sqrt{2+\sqrt{2+\ldots+\sqrt{2}}}$ konvergira. Izlazi da za $\alpha > 1$ zadatak nema smisla jer dobijemo imaginarne brojeve zbog $2-\sqrt{2+\sqrt{2+\ldots+\sqrt{2}}} < 0$. Za $\alpha < 1$ opšti član reda (*) ne teži 0, pa red u tom slučaju Ⓓ.
Za $\alpha = 1 \Rightarrow$ da $a_n \to 0$ pa ima smisla ispitivati konvergenciju. Prema prethodnom zadatku dati red se svodi na:

$$\sum_1^\infty (-1)^{n-1}\left(2\sin\frac{\pi}{2^{n+1}}\right)^\beta = \sum_1^\infty (-1)^{n-1}\left(\sin\frac{\pi}{2^n}\right)^\beta.$$

* Apsolutna konvergencija

$a_n = \left(\sin\frac{\pi}{2^n}\right)^\beta \sim \left(\frac{\pi}{2^n}\right)^\beta = \frac{\pi^\beta}{2^{\beta n}}$; $\Delta(koš) = \sqrt[n]{a_n} \approx \frac{1}{2^\beta}$.

Ⓓ $\frac{1}{2^\beta} > 1 \Rightarrow 2^\beta < 1 \Rightarrow 2^\beta < 2^0 \Rightarrow \beta < 0$ Ⓛ Ⓓ
Ⓚ $\frac{1}{2^\beta} < 1 \Rightarrow 2^\beta > 1 \Rightarrow 2^\beta > 2^0 \Rightarrow \beta > 0$ Ⓣ Ⓚ
} ili Limesom.

② Ispitati konvergenciju reda: $\sum_{1}^{\infty} \frac{\ln^{100} n}{n} \sin\frac{n\pi}{4}$.

koristimo Dirihleov kriterij: „Red $\sum_{n=1}^{\infty}$ konvergira ako $a_n \to 0$ monotono i ako $\sum_{k=1}^{n} b_k$ čini ograničen niz."

$a_n = \frac{\ln^{100} n}{n}$; $b_n = \sin\frac{n\pi}{4}$

$a_n = \frac{\ln^{100} n}{n} = \left| \begin{matrix} \ln n = t, \\ n = e^t \end{matrix} \right| = \frac{t^{100}}{e^t} \to 0, \ t \to \infty$

Trebamo dokazati da je f-ja monotono opadajuća $f(t) = \frac{t^{100}}{e^t}$ u okolini točke 0_+ kod $t \to \infty$.

$f'(t) = \frac{100t^{99}e^t - e^t \cdot t^{100}}{e^{2t}} < 0$ - mora biti $100t^{99}e^t < e^t t^{100} \,/ : e^t t^{99}$

$100t^{99}e^t < e^t t^{100} \,/ : e^t t^{99} \Rightarrow 100 < t$ za $t \gg$ ⇒ Ⓣ

Suma $\sum_{k=1}^{n} b_k = \sum_{k=1}^{n} \sin\frac{k\pi}{4} = \operatorname{Im} \sum_{k=1}^{n} e^{ik\frac{\pi}{4}} =$

$= \operatorname{Im}\left\{ e^{\frac{\pi}{4}} + e^{\frac{2\pi}{4}} + \ldots + e^{\frac{n\pi}{4}} \right\} = \operatorname{Im}\left\{ e^{\pi/4}\left(1 + e^{\pi/4} + e^{2\pi/4} + \ldots + e^{\frac{(n-1)\pi}{4}}\right)\right\} =$

$= \operatorname{Im}\left\{ e^{\frac{\pi}{4}}\left(\frac{1-e^{in\frac{\pi}{4}}}{1-e^{i\frac{\pi}{4}}}\right)\right\} = \operatorname{Im}\left\{ e^{\frac{\pi}{4}} \frac{(1-e^{in\frac{\pi}{4}})(1-e^{-\pi/4})}{(1-e^{i\pi/4})(1-e^{-i\frac{\pi}{4}})} \right\} =$

$= \operatorname{Im}\left\{ e^{\pi/4} \cdot \frac{1-e^{-\pi/4} - e^{in\frac{\pi}{4}} + e^{i(n-1)\pi/4}}{1-e^{i\pi/4} - e^{-i\frac{\pi}{4}} + 1} \right\} = \operatorname{Im}\left\{ \frac{e^{\pi/4} - 1 - e^{i(n+1)\pi/4} + e^{\frac{in\pi}{4}}}{2 - 2\cos\pi/4} \right\} =$

$= \operatorname{Im}\left\{ \frac{-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{(n+1)\pi}{4} - i\sin\frac{(n+1)\pi}{4} + \cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}}{2+2\cos\frac{\pi}{4}} \right\} =$

$= \left| \frac{\sin\frac{\pi}{4} - \sin\frac{(n+1)\pi}{4} + \sin\frac{n\pi}{4}}{2+2\cos\frac{\pi}{4}} \right| \leq \frac{3}{2+\sqrt{2}} \Rightarrow$

$\left| \sum_{k=1}^{n} \sin\frac{k\pi}{4} \right| \leq \frac{3}{2+\sqrt{2}}$ Sada po Dirihleovom kriteriju red <u>konvergira</u>.

Ⓩ Ispitati konvergenciju reda: $\sum_{n=1}^{\infty} \sin \pi\sqrt{n^2+k^2}$, $k \in N$.

$$a_n = \sin\pi\left(\sqrt{n^2+k^2}-n+n\right) = \sin\left(\pi\sqrt{n^2+k^2}-n\pi+n\pi\right) = \sin\left(\pi\sqrt{n^2+k^2}-n+n\pi\right)$$

$$= \cos n\pi \sin\frac{n^2+k^2-n^2}{\sqrt{n^2+k^2}+n}\pi + \sin n\pi \cos\left(\sqrt{n^2+k^2}-n\right)\pi =$$

$$= \underbrace{\cos n\pi}_{(-1)^n}\cdot\sin\frac{k^2\pi}{\sqrt{n^2+k^2}+n} \sim (-1)\frac{\pi k^2}{2n}.$$

$$\left.\begin{array}{l} a_n = \frac{\pi k^2}{2n} \to 0,\ n \to +\infty \\ a_n > a_{n+1} \Rightarrow \frac{\pi k^2}{2n} > \frac{\pi k^2}{2(n+1)} \quad Ⓣ \end{array}\right\} \begin{array}{l}\text{Aps. Ⓓ} \\ \text{Usl. Ⓚ}\end{array}$$

Ⓩ Ispitati konvergenciju reda $\sum_{n=1}^{\infty}\left(\frac{1+\cos n}{2+\cos n}\right)^{2n-\ln n}$

Iskoristimo $\Delta(\text{koši}) = \sqrt[n]{a_n} \Rightarrow$

$$\left(\frac{1+\cos n}{2+\cos n}\right)^{(2n-\ln n)\cdot\frac{1}{n}} = \left(\frac{1+\cos n}{2+\cos n}\right)^{2-\frac{\ln n}{n}} = \left(\frac{1+\cos n}{2+\cos n}\right)^2 \cdot \frac{1}{\left(\frac{1+\cos n}{2+\cos n}\right)^{\frac{\ln n}{n}}}$$

Nađimo $\lim_{(n)} \sqrt[n]{a_n}$!

$$\lim_{(n)}\left(\frac{1+\cos n}{2+\cos n}\right)^2\cdot\frac{1}{\left(\frac{1+\cos n}{2+\cos n}\right)^{\frac{\ln n}{n}}} \Rightarrow \lim_{(n)}\frac{\ln n}{n} = \left|\begin{array}{l}\ln n = t, \\ e^t = n\end{array}\right| = \lim_{t\to+\infty}\frac{t}{e^t} = 0.$$

$$= \lim_{(n)}\left(\frac{1+\cos n}{2+\cos n}\right)^2 \le \left(\frac{2}{3}\right)^2 \le \frac{4}{9}$$

kako je $\frac{4}{9} < 1 \Rightarrow$ red Ⓚ.

Ⓩ Ispitati red čiji je opšti član $a_n = \log_{b^n}\left(1+\frac{\sqrt[n]{a}}{n}\right)$, $a>0$, $b>0$. Odavde dobivamo da je:

$$b^{na_n} = 1+\frac{\sqrt[n]{a}}{n} \quad /\odot \ln$$

$$na_n \ln b = \ln\left(1+\frac{\sqrt[n]{a}}{n}\right)\text{. tj. } a_n = \frac{\ln\left(1+\frac{\sqrt[n]{a}}{n}\right)}{n\ln b} = \left|\begin{array}{c}\ln(1+x)\sim x \\ x\to 0\end{array}\right| =$$

$$\sim a_n \approx \frac{\sqrt[n]{a}}{n^2\ln b} \sim \frac{1}{n^2\ln b}$$

$$\Delta(\log) = \frac{2\ln\ln b}{\ln n},\ \lim_{(n)}\frac{2\ln\ln b}{\ln n} \to 0 < 1 \Rightarrow \text{red Ⓓ}.$$

② Ispitati apsolutnu i uslovnu konvergenciju reda: $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n^\alpha \ln^p n}$.

Uslovna konvergencija: $a_n = \frac{1}{n^\alpha \ln^p n} = \left| \begin{matrix} \ln n = t \\ e^t = n \end{matrix} \right| = \frac{1}{e^{\alpha t} t^p}$.

$a_n = \frac{1}{e^{\alpha t} t^p}$. Prema L. kriteriju moraju da budu zadovoljena dva uslova: $1^\circ \lim_{(n)} a_n \to 0$

$2^\circ\ a_n > a_{n+1}$.

$a_n = \frac{1}{e^{\alpha t} t^p}$ $\lim_{(n)} \frac{1}{e^{\alpha t} t^p}$ $= +\infty,\ \alpha < 0,\ \forall p$ | Usl. i Aps. ⓓ

$+\infty,\ \alpha = 0,\ p < 0$ | Usl. i Aps. ⓓ

$1,\ \alpha = 0,\ p = 0$ | Usl. i Aps. ⓓ

za $\alpha < 0$ imali bi:

$a_n = \frac{e^{\alpha t}}{t^p}$ za $p > 0$ ovo teži u $+\infty$; za $p < 0$ $\longrightarrow +\infty$; za $p = 0$ $\longrightarrow +\infty$ $\Big\}$ $\forall p \to +\infty$.

za $\alpha > 0$ imali bi:

$a_n = \frac{1}{e^{\alpha t} t^p}$ za $p > 0 \longrightarrow 0$; za $p = 0 \longrightarrow 0$; za $p < 0$ imali bi: $a_n = \frac{t^p}{e^{\alpha t}} \to 0$ $\Big\}$ $\forall p \to 0$.

za $\alpha = 0$ imali bi:

$a_n = \frac{1}{t^p}$ za $p > 0 \longrightarrow 0$; za $p < 0 \longrightarrow +\infty$; za $p = 0 \longrightarrow 1$

Ispitajmo drugi uslov Lajbnicovog kriterija: kako je $a_n = \frac{1}{e^{\alpha x} x^p}$ potražimo prvi izvod.

$$f'(x) = \frac{-1(\alpha e^{\alpha x} x^p + p \cdot x^{p-1} \cdot e^{\alpha x})}{(e^{\alpha x} \cdot x^p)^2}$$

kako niz mora biti opadajući to mora biti ispunjen uslov $f'(x) < 0$,

$f'(x) < 0$ tj. $\alpha e^{\alpha x} \cdot x^p + p x^{p-1} e^{\alpha x} > 0 \quad / : e^{\alpha x} > 0$

$\alpha x^p + p x^{p-1} > 0$

$\alpha x^p > -p x^{p-1}$ Očito je ovo zadovoljeno jer je za $p>0$ lijeva strana pozitivna, a desna negativna. Dalje, dobivamo da je $\alpha x > -p$ što važi i za $p<0$. Dakle, niz je monotono opadajući $\forall p$. Odovde i iz Lajbnicovog kriterija izlazi da niz uslovno konvergira za $\alpha > 0$ i $\forall p$.

Apsolutna konvergencija: $a_n = \frac{1}{n^\alpha \ln^p n}$

$$\Delta(\log) = \frac{\ln\left(\frac{1}{n^\alpha \ln^p n}\right)^{-1}}{\ln n} = \frac{\ln n^\alpha \ln^p n}{\ln n} = \frac{\ln n^\alpha + \ln \ln^p n}{\ln n} =$$

$$= \alpha \frac{\ln n}{\ln n} + p \frac{\ln \ln n}{\ln n} = \alpha + p \frac{\ln \ln n}{\ln n} = \left| \begin{array}{l} \ln n = x, \\ e^x = n \end{array} \right| = \alpha + p \frac{\ln x}{x} =$$

$$= \left| \begin{array}{l} \ln x = t, \\ e^t = x \end{array} \right| = \alpha + p \frac{t}{e^t} = \underline{\alpha + p \frac{t}{e^t}}$$

$$\lim_{t \to \infty} \left(\alpha + \underbrace{p \frac{t}{e^t}}_{\to 0}\right) = \alpha ,$$

za $\alpha > 1$ red Aps. (K)

$\alpha < 1$ red Aps. (D)

$\alpha = 1$ koristimo Bertrandov kriterij!

$a_n = \frac{1}{n \ln^p n}$

$$\Delta(BERT): \left[n \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right] \ln n \quad \begin{array}{ll} > 1 & (K) \\ < 1 & (D) \end{array}$$

$\frac{a_n}{a_{n+1}} = \frac{(n+1) \ln^p (n+1)}{n \ln^p n}$ pa je $\Delta(BERT) = \left[n \left(\frac{(n+1) \ln^p (n+1)}{n \ln^p n} \right) - 1 \right] \cdot \ln n =$

$$= \left[\frac{(n+1) \ln^p (n+1) - n \ln^p n}{\ln^p n} - 1 \right] \cdot \ln n = \frac{(n+1) \ln^p (n+1) - n \ln^p n - \ln^p n}{\ln^p n} \cdot \ln n =$$

$$= \frac{(n+1) \ln^p (n+1) - (n+1) \ln^p n}{\ln^p n} \cdot \ln n = (n+1) \cdot \left[\frac{\ln^p (n+1) - \ln^p n}{\ln^p n} \right] \ln n =$$

$$= (n+1)\ln n\left[\left(\frac{\ln(n+1)}{\ln n}\right)^p - 1\right] = (n+1)\ln n\left[e^{\left(\ln\frac{\ln(n+1)}{\ln n}\right)^p} - 1\right] = \left|\begin{matrix} e^u - 1 \sim u \\ u \to 0\end{matrix}\right| =$$

$$\sim (n+1)\ln n \cdot p \ln\underbrace{\frac{\ln(n+1)}{\ln n}} = \left|\begin{matrix}\ln u \sim u-1 \\ u \to 1\end{matrix}\right| \sim (n+1)\ln n \cdot p\left(\frac{\ln(n+1)}{\ln n} - 1\right) =$$

$$= p(n+1)\ln n\,\frac{\ln(n+1) - \ln n}{\ln n} = p(n+1)\ln\left(\frac{n+1}{n}\right) = p\underbrace{\ln\left(\frac{n+1}{n}\right)^{n+1}}_{\to 1} = p$$

Dakle, $\Delta(BERT) = p > 1$ Ⓚ, $p < 1$ Ⓓ, $p = 1 \to$ treba ispitati!

$*$

$$\frac{1}{n} > \ln(n+1) - \ln n \quad \Big/ \cdot \frac{1}{\ln n} \qquad (1)$$

$$\frac{1}{n\ln n} > \frac{\ln(n+1) - \ln n}{\ln n} \qquad (2)$$

$$\frac{1}{n\ln n} > \frac{\ln(n+1)}{\ln n} - 1 \quad (*) \quad (3)$$

$\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$ Iz nejednakosti $\frac{1}{n} > \ln(1+\frac{1}{n})$ imamo:

$\ln\frac{\ln(n+1)}{\ln n} < \frac{\ln(n+1)}{\ln n} - 1$. Ovo i $(*)$ daju:

$\frac{1}{n\ln n} > \frac{\ln(n+1)}{\ln n} - 1 > \ln\frac{\ln(n+1)}{\ln n}$ tj. $\frac{1}{n\ln n} > \ln\frac{\ln(n+1)}{\ln n}$ i

dalje je: $\frac{1}{n\ln n} > \ln\ln(n+1) - \ln\ln n$. Stavljajući $n = 2$,

do n dobijemo:

$$\left.\begin{matrix}\frac{1}{2\ln 2} > \ln\ln 3 - \ln\ln 2 \\ \frac{1}{3\ln 3} > \ln\ln 4 - \ln\ln 3 \\ \vdots \\ \frac{1}{n\ln n} > \ln\ln(n+1) - \ln\ln n\end{matrix}\right\} \oplus$$

$S_n = \sum\limits_{k=2}^{n}\frac{1}{k\ln k}$ je veće od $\underbrace{\ln\ln(n+1) - \ln\ln 2}_{\to+\infty,\ n\to+\infty}$.

pa je $S_n = +\infty$..

To jest: $S_n = \sum\limits_{k=2}^{n}\frac{1}{k\ln k} > \ln\ln(n+1) - \ln\ln 2$ i kada $n \to +\infty$

tada i $\ln\ln(n+1) - \ln\ln 2 \to +\infty$ pa je $\lim\limits_{(n)} S_n = +\infty$ i

suma reda je $+\infty$ za $p = 1$ i time je red konačno ispitan.

Ⓩ Ispidati red s opštim članom : $a_n=\left(\cos\frac{1}{\sqrt{n}}\right)^{n^2}$.

Košijev kriterij korjena je najprikladniji :

$$\sqrt[n]{a_n}=\left(\cos\frac{1}{\sqrt{n}}\right)^{n^2\cdot\frac{1}{n}}=\left(\cos\frac{1}{\sqrt{n}}\right)^{n}$$

$$L=\lim_{(n)}\left(\cos\frac{1}{\sqrt{n}}\right)^{n}\ /\odot\ln\ \Rightarrow\ \ln L=\lim_{(n)} n\cdot\ln\left(\cos\frac{1}{\sqrt{n}}\right)=$$

$$=\left|\begin{matrix}\ln u\sim u-1,\\ u\to 1\end{matrix}\right|\Rightarrow \ln L=\lim_{(n)} n\left(\cos\frac{1}{\sqrt{n}}-1\right)=\lim_{(n)}\left(-2n\sin^2\frac{1}{2\sqrt{n}}\right)=$$

$$=-2\lim_{(n)} n\sin^2\frac{1}{2\sqrt{n}}=\left|\begin{matrix}\sin\alpha\sim\alpha,\\ \alpha\to 0\end{matrix}\right|=-2\lim_{(n)} n\cdot\frac{1}{4n}=-\frac{1}{2}$$

$$\ln L=-\frac{1}{2}\Rightarrow L=e^{-\frac{1}{2}}=\frac{1}{\sqrt{e}}<1 \text{ pa red (k)}.$$

Ⓩ Ispitati konvergenciju reda : $\sum_{n=1}^{\infty}\arcsin\frac{1}{\sqrt{n}}\cdot\cos\frac{1}{n^2}(-1)^n$.

$a_n=\arcsin\frac{1}{\sqrt{n}}\cos\frac{1}{n^2}(-1)^n$ upotrijebimo smjenu: $\cos x\sim 1, x\to 0$ i $\arcsin x\sim x, x\to 0$.

$$a_n\cong\frac{1}{\sqrt{n}}(-1)^n=\frac{(-1)^n}{n^{1/2}}$$

Uslovno (Lajbnic): 1° $\lim\frac{1}{n^{1/2}}\to 0$ (T)

$a_n=\frac{1}{\sqrt{n}}>0$ $\quad 2^\circ$ $a_n>a_{n+1}\Rightarrow\frac{1}{\sqrt{n}}>\frac{1}{\sqrt{n+1}}$ (T) $\Big\}$ Usl. (k)

Apsolutna :

$$a_n=\frac{1}{\sqrt{n}}\Rightarrow\Delta(\log)=\frac{\ln\frac{1}{a_n}}{\ln n}=\frac{\frac{1}{2}\ln n}{\ln n}=\frac{1}{2}<1\Rightarrow \text{Aps. (D)}$$

Ⓩ Ispitaj konvergenciju reda : $\sum_{1}^{\infty}\frac{(-1)^{[\sqrt{n}]}}{n}$ gdje je $[n]$ prvi cio broj koji nije veći od n.

Taj red izgleda ovako:

$$S_0=-\left(1+\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)-\left(\frac{1}{9}+\frac{1}{10}+\dots+\frac{1}{15}\right)+\dots$$

Sada je to red : $\sum_{1}^{\infty}(-1)^n A_n$ gdje je :

$$A_n=\frac{1}{n^2}+\frac{1}{n^2+1}+\dots+\frac{1}{n^2+(n-1)}.$$

$$\frac{2n+1}{n^2+(n-1)} \le A_n < \frac{2n+1}{n^2} \quad / \cdot \lim_{(n)} \quad (*)$$

$1^\circ \ \lim_{(n)} A_n \to 0$

$2^\circ \ A_{n+1} \le \frac{2n+3}{(n+1)^2}$ Dokažimo sada $\frac{2n+3}{(n+1)^2} < \frac{2n+1}{n^2+(n-1)}$

odakle slijedi $A_{n+1} < A_n$

$$(2n+3)(n^2+n-1) < (2n+1)(n^2+2n+1)$$

$$2n^3+2n^2-2n+3n^2+3n-3 < 2n^3+4n^2+2n+n^2+2n+1$$

$$\cancel{2n^3}+\cancel{5n^2}+n-3 < \cancel{2n^3}+\cancel{5n^2}+4n+1$$

$\underline{n-3 < 4n+1}$ (T) Prema L. red (K).

predavanja

Funkcije

$y = 2\sin x + \cos 2x$

RJ:

1° $D(f)$: F-ja $\sin x$ i f-ja $\cos x$ definisane su na cijelom skupu $\mathbb{R}$ pa je i naša f-ja definisana $\forall x \in \mathbb{R}$.

2° Periodičnost: kada imamo zbir dvije periodične f-je tada će rezultantna f-ja biti periodična ako količnik perioda tih f-ja bude cio broj.

$$\left.\begin{array}{l} \sin x \Rightarrow T_1 = 2\pi \\ \cos 2x \Rightarrow T_2 = \frac{2\pi}{2} = \pi \end{array}\right\} \frac{T_1}{T_2} = \frac{2\pi}{\pi} = 2 \in Q$$

Period je sada $T = \max\{T_1, T_2\}$, a to je $\underline{T = 2\pi}$.

3° Nule:

$$y = 0 \Rightarrow 2\sin x + \cos 2x = 0$$

$$2\sin x - 2\sin^2 x + 1 = 0 \quad / \cdot -\tfrac{1}{2}$$

$$\sin^2 x - \sin x - \tfrac{1}{2} = 0 \Rightarrow |\sin x = x|$$

$$x^2 - x - \tfrac{1}{2} = 0, \quad x_{1,2} = \frac{1 \pm \sqrt{1+2}}{2} = \frac{1 \pm \sqrt{3}}{2}$$

$$(\sin x)_{1,2} = \frac{1 \pm \sqrt{3}}{2}$$

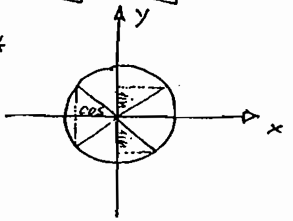
* $\cos 2x = \cos^2 x - \sin^2 x$
$\cos^2 x = 1 - \sin^2 x$
$\cos 2x = 1 - 2\sin^2 x$

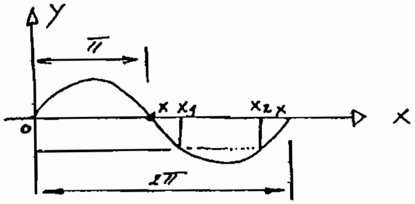
kako je $\sin x$ ograničena f-ja uzimamo rješenje koje odgovara tim granicama.

$|\sin x| \leq 1$ tj., $-1 \leq \sin x \leq 1$. Nama odgovara

$\sin x = \frac{1-\sqrt{3}}{2} \Rightarrow$ $\boxed{x = \arcsin 0{,}365}$ → Ovo rješenje daje dva rješenja:

*

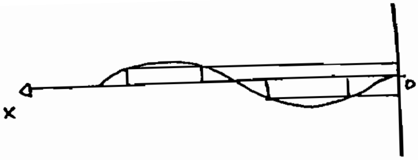




$\sin x = \frac{1-\sqrt{3}}{2} \cong -0{,}365 \Rightarrow \underline{x = -\arcsin 0{,}365}$

↳ daje dva rješenja:

$x_1 = \pi + x = \pi + \arcsin(-0{,}365)$
$x_2 = 2\pi - x = 2\pi - \arcsin(-0{,}365)$



4° Znak :

$f(x) > 0 \Rightarrow \sin^2 x - \sin x - \frac{1}{2} > 0 \Rightarrow (0 < x < x_1 \wedge x_2 < x < 2\pi)$

$f(x) < 0 \Rightarrow \sin^2 x - \sin x - \frac{1}{2} < 0 \Rightarrow (x_1 < x < x_2)$

5° Izvodi :

$f'(x) = 2\cos x - 2\sin 2x = 2(\cos x - \sin 2x) = 2\cos x(1 - 2\sin x)$

$f''(x) = -2\sin x(1 - 2\sin x) - 4\cos^2 x = -2\sin x + 4\sin^2 x - 4\cos^2 x =$

$= -2\sin x + 4\sin^2 x - 4(1 - \sin^2 x) = 8\sin^2 x - 2\sin x - 4 =$

$= 2(4\sin^2 x - \sin x - 2).$

$f'(x) = 0$ za : $\cos x(1 - 2\sin x) = 0 \Rightarrow \cos x = 0 \Rightarrow x_1 = \frac{\pi}{2}, x_2 = \frac{3\pi}{2}$

$1 - 2\sin x = 0 \Rightarrow x_3 = \frac{\pi}{6}, x_4 = \frac{5\pi}{6}$

$f''(x) = 0$ za : $4\sin^2 x - \sin x - 2 = 0$

$\sin_{1,2} x = \frac{1 \pm \sqrt{33}}{8}$ (upada u granice)

$\sin x_1 \approx 0{,}84 \Rightarrow x_3 = 57°, x_4 = 123°$

$\sin x_2 \approx -0{,}6 \Rightarrow x_5 = 217°, x_6 = 323°$

$f''(x) > 0 \Rightarrow (\sin x < \sin x_2 \cup \sin x > \sin x_1) \Rightarrow$

$\Rightarrow (270° \leq x < 323° \cup 217° < x \leq 270°) \cup (57° < x \leq \frac{\pi}{2} \vee \frac{\pi}{2} \leq x < 123°)$

$\Rightarrow x \in (57°, 123°) \cup (217°, 323°)$

$f''(x) < 0 \Rightarrow (0 \leq x < 57° \vee 123° < x < 217° \vee 323° < x \leq 360°)$

6° Prevojne tačke :

Drugi izvod mijenja znak u sve četiri svoje nule, tj., u nulama f-je $f''(x)$ f-ja $f(x)$ ima prevojne tačke.

$f''(\frac{\pi}{6}) < 0 \Rightarrow Y_{max} = f(\frac{\pi}{6}) = \frac{3}{2}$

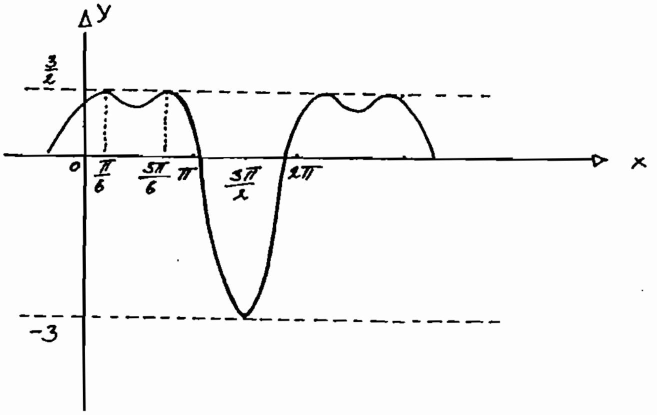
$f''(\frac{5\pi}{6}) < 0 \Rightarrow Y_{max} = f(\frac{5\pi}{6}) = \frac{3}{2}$

$f''(\frac{\pi}{2}) = 2 > 0 \Rightarrow y_{min} = f(\frac{\pi}{2}) = 1$

$f''(\frac{3\pi}{2}) = 6 > 0 \Rightarrow y_{min} = f(\frac{3\pi}{2}) = -3.$

7° Grafik f-je je konkavan u intervalima: (x_3, x_4) i (x_5, x_6) a konveksan u $[0, x_3)$, (x_4, x_5) i $(x_6, 360°]$.

8° $\lim\limits_{x \to \pm\infty} \sin x$ = ne postoji ⇒ da ne postoje limesi: $\lim\limits_{x \to \pm\infty} f(x)$ i $\lim\limits_{x \to \pm\infty} \frac{f(x)}{x}$ tako da grafik nema ni horizontalnih ni vertikalnih (jer je ograničen grafik) ni kosih asimptota.



② Ispitati tok i nacrtati grafik f-je:

$f(x) = 2x - 3\sqrt[3]{x^2}$

RJ:

1° $D(f)$: $\forall x \in R$.

2° Nije simetrična: $f(-x) = -2x - 3\sqrt[3]{x^2}$

3° Izvodi:

$y = 2x - 3\sqrt[3]{x^2} = 2x - 3x^{\frac{2}{3}}$

$y' = (2x - 3x^{\frac{2}{3}})' = 2 - 3 \cdot \frac{2}{3} x^{\frac{2}{3} - \frac{3}{3}} = 2 - 2x^{-\frac{1}{3}} = 2 - \frac{2}{x^{\frac{1}{3}}}$

$y' = 2 - \frac{2}{x^{\frac{1}{3}}}$, $y' = 0$ za $x = 1$ – STACIONARNA TAČKA

$D(f')$: $x \neq 0$.

$y'' = \left(2 - \frac{2}{x^{\frac{1}{3}}}\right)' = 0 + 2\cdot\frac{1}{3}x^{-\frac{4}{3}} = \frac{2}{3x^{4/3}}$

$y'' = \frac{2}{3x^{4/3}} \qquad D(f'): x \neq 0.$

4° Nule :

$y = 0$ za $2x - 3\sqrt[3]{x^2} = 0 \Rightarrow 2x = 3\sqrt[3]{x^2} \,/^3 \Rightarrow 8x^3 = 27x^2 \Rightarrow x = 0$

$x = \frac{27}{8}$

5° Znak :

$f(x) > 0 \Rightarrow 8x^3 - 27x^2 > 0 \Rightarrow x > \frac{27}{8}$

$f(x) < 0 \Rightarrow x < \frac{27}{8} \wedge x \neq 0.$

6° Intervali monotonosti :

$(-\infty, 0)$, $(0, +\infty)$

7° Extrem :

$x = 1$

$y''(x) = \frac{2}{3x^{4/3}}$ Extrem: $T.(1,-1)$.

$y''(1) = \frac{2}{3} > 0 \Rightarrow$ minimum

8° Ispitajmo definiciono područje :

$$\begin{cases} \lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}(2x - 3\sqrt[3]{x^2}) = \lim\limits_{x\to+\infty} x\left(2 - \frac{3}{\sqrt[3]{x}}\right) = +\infty \\ \lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x\left(2 - \frac{3}{\sqrt[3]{x}}\right) = -\infty \end{cases}$$

$$\left.\begin{cases} \lim\limits_{x\to 0_+} f(x) = \lim\limits_{x\to 0+}(2x - 3\sqrt[3]{x^2}) = 0 \\ \lim\limits_{x\to 0_-} f(x) = \lim\limits_{x\to 0-}(2x - 3\sqrt[3]{x^2}) = 0 \end{cases}\right\}$$

$L_1 = L_2 \Rightarrow$ f-ja je neprek. u tački $x = 0$.

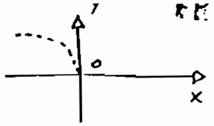
$$\left.\begin{cases} \lim\limits_{x\to 1_+} f(x) = (2-3) = -1 \\ \lim\limits_{x\to 1_-} f(x) = (2-3) = -1 \end{cases}\right\}$$

$L_1 = L_2 \Rightarrow$ f-ja je neprekidna u tački $x = 1$.

kako f-ja nema izvoda u tački $x = 0$, a tu je neprekidna i ima neku vrijednost moguće je da

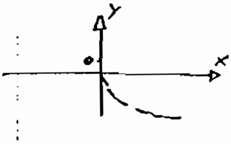
$$\lim_{x\to 0-} f'(x) = \lim_{x\to 0-} \left(2 - \frac{2}{x^{1/3}}\right) = +\infty \qquad tg\,\alpha = +\infty,\ \alpha = 90°$$

$$\lim_{x\to 0+} f'(x) = \lim_{x\to 0+} \left(2 - \frac{2}{x^{1/3}}\right) = -\infty \qquad tg\,\alpha = -\infty,\ \alpha = 270°$$



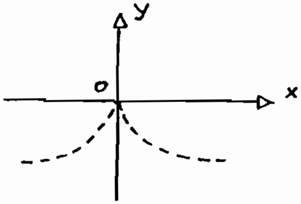
$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{2x - 3\sqrt[3]{x^2}}{x} = -\infty$$

$$\lim_{x\to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{2x - 3\sqrt[3]{x^2}}{x} = +\infty$$



Ne znamo da li je ovo tačno na crtežima jer ne znamo kakav je ulaz u tačku (odozdo ili odozgo). Radi toga gledamo konkavnost i konveksnost f-je.

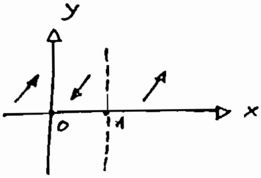
$y''(x) = \frac{2}{3x^{4/3}} > 0 \Rightarrow$ f-ja konveksna ∪, pa je crtež:



9° Tok f-je:

$y' > 0 \Rightarrow 1 - \frac{1}{x^{1/3}} \Rightarrow \quad x > 1 \lor x < 0$ „↗" raste

$y' < 0 \Rightarrow 1 - \frac{1}{x^{1/3}} < 0 \Rightarrow \quad 0 < x < 1$ – „↙" opada



10° Prevojne tačke

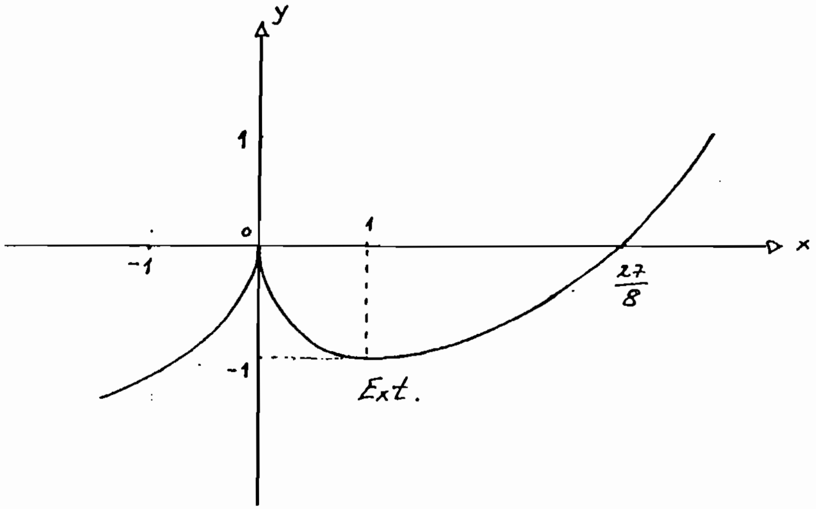
$y'' \neq 0 \ \forall x \in R$ pa nemamo prevojnih tačaka.

11° Asimptote:

- Nema vertikalnih jer je neprekidna na cijelom int.
- Nema horizontalnih: $\lim_{x\to\pm\infty} f(x) = \pm\infty$
- Nema ni kosih jer je

$\lim_{x\to\pm\infty} \frac{f(x)}{x} = k = 2$ a $n = \lim_{x\to\pm\infty} (f(x) - 2x) = -\infty.$

12° Grafik:



③ Nacrtati grafik funkcije: $\frac{x^4+1}{x^2} = f(x)$

1° $D(f)$: $x \neq 0$, $x \in \mathbb{R}$.

2° Simetrija:

$f(-x) = \frac{x^4+1}{x^2} = f(x) \Rightarrow$ parno simetrična f-ja,
simetrična u odnosu na y osu.

3° Nule : $\frac{x^4+1}{x^2} = 0$, $x^4+1 \neq 0$ – nema nula.

4° Izvodi :

$$y' = \left(\frac{x^4+1}{x^2}\right)' = \frac{4x^3 \cdot x^2 - 2x(x^4+1)}{x^4} = \frac{4x^5 - 2x^5 - 2x}{x^4} = \frac{2x^5 - 2x}{x^4} =$$

$$= \frac{2x(x^4-1)}{x^4}$$

$$y' = \frac{2(x^4-1)}{x^3} \Rightarrow D(f'(x)): \quad x \neq 0$$

$y' = 0$ za $x^4 - 1 = 0 \Rightarrow x = \pm 1$

$\left.\begin{matrix} x = -1 \\ x = 1 \end{matrix}\right\}$ ST. I vrste. + $\boxed{x = 0}$.

$$y'' = \left(\frac{2x^4-2}{\dots}\right)' = \frac{8x^3 \cdot x^3 - 3x^2(2x^4-2)}{\dots} = \frac{6}{\dots} \Rightarrow$$

$\Rightarrow \quad D(f''(x)): \quad x \neq 0.$ ST. II vrste je $x=0$ jer drugi izvod nije definisan u ovoj tački. Ostalih stac. tč. nema.

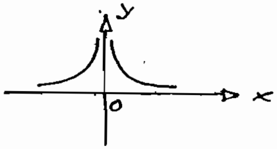
$y''=0$ ni za jedno $x \in R$.

5° Ispitajmo definiciono područje:

$(-\infty,0) \cup (0,+\infty)$

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\frac{x^4+1}{x^2}\right) = +\infty$$
$$\lim_{x\to-\infty} f(x) = +\infty$$
} Nema horizontalnih asimptota.

$$\lim_{x\to 0_-}\left(\frac{x^4+1}{x^2}\right) = +\infty \Rightarrow \text{V.AS.}$$
$$\lim_{x\to 0_+}\left(\frac{x^4+1}{x^2}\right) = +\infty \Rightarrow \text{V.AS.}$$
} $\Rightarrow$



6° Tok f-je.

Potrebno je ispitati prvi izvod.

$$y' = \frac{2(x^4-1)}{x^3} = \frac{2((x^2)^2-1^2)}{x^3} = \frac{2\cdot(x^2-1)(x^2+1)}{x^3} = \frac{(x+1)(x-1)(x^2+1)\cdot 2}{x^3}$$

Na znak utiču samo svi članovi koji mijenjaju znak. To su: $(x+1), (x-1), x^3$. Formiramo tabelu:

	$-\infty$ … -1	-1 … 0	0 … 1	1 … $+\infty$
$x+1$	−	+	+	+
x^3	−	−	+	+
$x-1$	−	−	−	+
	⊖	⊕	⊖	⊕
	↙	↗	↙	↗

$(-\infty,-1)$ ↙

$(-1,0)$ ↗

$(0,1)$ ↙

$(1,+\infty)$ ↗

7° Ekstremi:

ST. I vrste su tačke $\left.\begin{matrix} x=-1 \\ x=1 \end{matrix}\right\}$

$y''=\frac{6}{x^4}$ pa imamo dva minimuma: Ext.1 $(-1,2)$

Ext.2 $(1,2)$.

8° Asimptote:

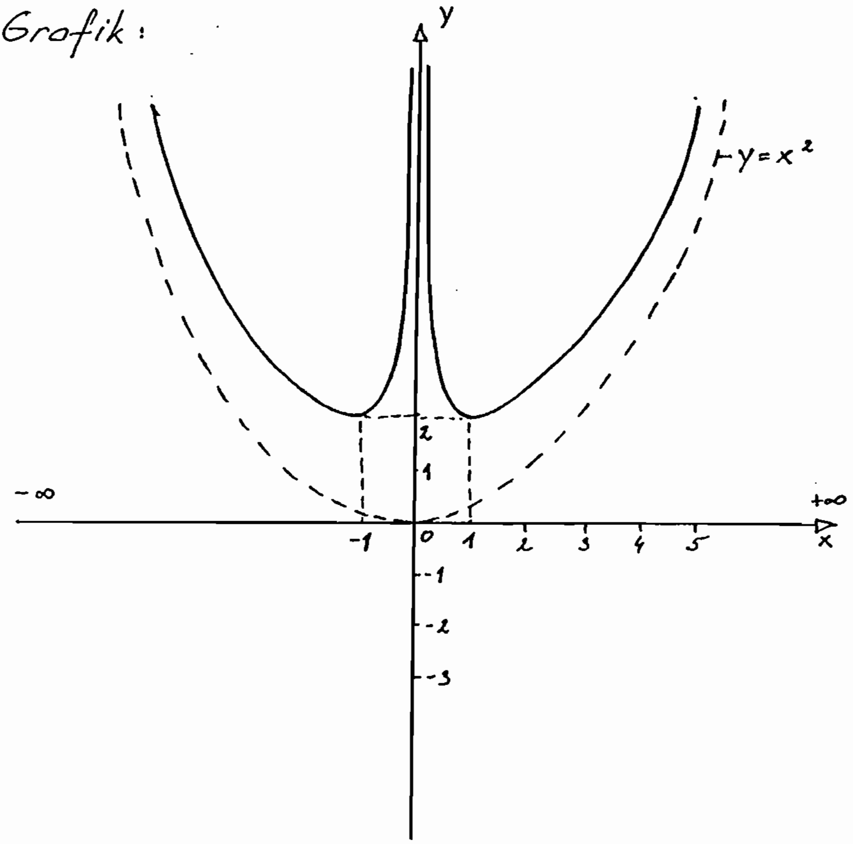
$\lim\limits_{x\to\pm\infty} \frac{f(x)}{x} = \lim\limits_{x\to\pm\infty} \frac{x^4+1}{x^3} = +\infty$ Nema kosih asimptota.

* kada tražimo $\lim\limits_{x\to\pm\infty} \frac{f(x)}{x}$ tada u stvari tražimo u kojoj mjeri $f(x)$ simulira f-ju $y=x$. Ako dobijemo konačan broj taj je ta mjera i predstavlja koeficijent uz x.

$\lim\limits_{x\to\pm\infty} \frac{f(x)}{x^2} = \lim\limits_{x\to\pm\infty} \frac{\frac{x^4+1}{x^2}}{x^2} = 1+\frac{1}{x^4} = 1$, pa f-ja:

$y=\frac{x^4+1}{x^2}$ simulira krivu $y=1\cdot x^2 = x^2$.

9° Grafik:



② Nacrtati f-ju: $y = f(x)$ ako je: $(y-x)^2 = x^3$.

RJ: $(y-x)^2 = x^3 \Rightarrow y - x = \pm\sqrt{x^3} \Rightarrow$ da imamo dvije f-je, a ne dvoznačnu.

$$y = \pm\sqrt{x^3} + x$$

$$\Downarrow$$

$$y_1 = \sqrt{x^3} + x$$

$$y_2 = -\sqrt{x^3} + x$$

1° $y = \sqrt{x^3} + x$, $D(f): x \geq 0$

2° Nule: $y = 0$ za $-x = \sqrt{x^3} \; /^2$

$x = x^3$ — $x = 0$

$x = 1$ - je prividna nula jer se kvadriranjem izgubi jedan minus.

3° Simetrija: $f(-x) = \sqrt{-x^3} - x$ - nije simetrična.

4° Izvodi:

$y = \sqrt{x^3} + x \Rightarrow y' = \frac{3}{2}\sqrt{x} + 1$, $x \geq 0$

$y' = 0$ - nije nikad,

$y'' = \frac{3}{4} \cdot \frac{1}{\sqrt{x}} = \frac{3}{4\sqrt{x}}$, $x > 0$

$y'' = 0$ - nije nikad

zbog toga nema ST. I, II pa nema ekstrema ni prevoja.

5° Ispitajmo definiciono područje:

$f'(x) > 0$ - rastuća ↗

$f''(x) > 0$ - „U" konkavna

$y'(0) = 1$

$$y'(0+) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{\sqrt{x^3} + x}{x} = 1 \Rightarrow tg\,\alpha = 1.$$

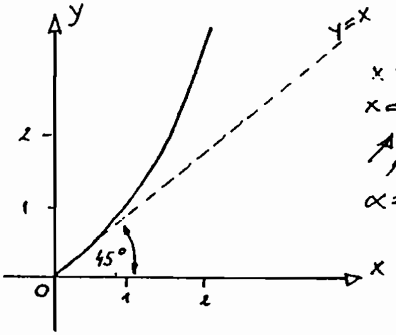
$\alpha = 45°$ - ugao izlaza.

6° Asimptote :

- Nema horizontalne
- Nema vertikalne

$\lim_{x\to\pm\infty} \frac{\sqrt{x^3}+x}{x} = k \Rightarrow k=\pm\infty$ – nema kosih asm.

7° Grafik:



$x \geq 0$
$x=0$ – nula f-je
↗, ∪
$\alpha = 45°$

Druga f-ja:

$y = x - \sqrt{x^3}$

1° $D(f)$: $x \geq 0$

2° Nule: $x = \sqrt{x^3} \Rightarrow x^2 = x^3 \Rightarrow x=0,\ x=1$

3° $f(-x) = -x - \sqrt{-x^3}$ – nije simetrična

4° Izvodi:

$y' = 1 - \frac{3\sqrt{x}}{2} \Rightarrow D(f'(x))$: $x \geq 0$

$f'(x) = 0$ za $\frac{3\sqrt{x}}{2} = 1 \Rightarrow 3\sqrt{x} = 2$

$\Rightarrow \sqrt{x} = \frac{2}{3} \Rightarrow x = \frac{4}{9}$ – STAC. TAČKA I vrste

$y'' = -\frac{3}{4\sqrt{x}} \Rightarrow D(f''(x))$: $x > 0$ $\quad y'' \neq 0.$

5° Extremi:

$f''\left(\frac{4}{9}\right) = -\frac{3}{4\sqrt{\frac{4}{9}}} < 0 \Rightarrow$ maximum $\quad$ Ext. $\left(\frac{4}{9}, \frac{4}{27}\right)$

6° Tok f-je: $\left(0, \frac{4}{9}\right)$ $\left(\frac{4}{9}, 1\right)$ $(1, +\infty)$

↗ ↙ ↗

7° $y''(x) = -\frac{3}{4\sqrt{x}} < 0 \Rightarrow$ „∩"

8° Asimptote:

$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \left(x - x^{\frac{3}{2}}\right) = -\infty$ nema H.AS.

E.H.

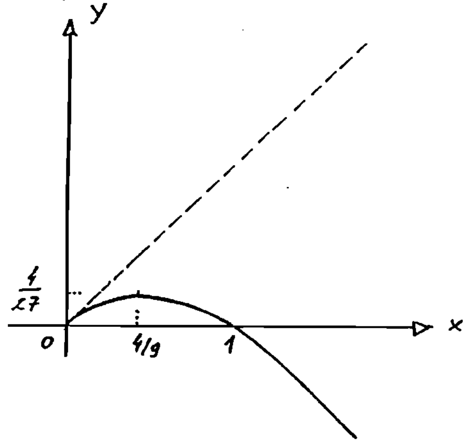
$k = \lim_{x\to+\infty} \frac{x-\sqrt{x^3}}{x} = 1-\sqrt{x} = -\infty$ nema K.AS.

9° Ugao ulaska u nulu:

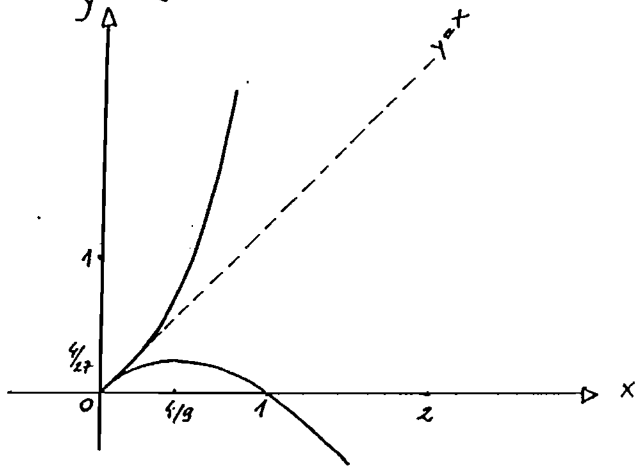
$\lim_{x\to0_+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to0_+} \frac{x-\sqrt{x^3}}{x} = \lim_{x\to0_+}(1-\sqrt{x}) = 1$

$\operatorname{tg}\alpha = 1 \Rightarrow \alpha = 45^\circ$

10° Grafik:



11° Grafik cijele f-je:



(2) Nacrtati f-ju: $g(x) = f^2(x) = x^2(8-x^2)$.

RJ: $y = \pm\sqrt{x^2(8-x^2)} = \pm\sqrt{8x^2-x^4}$ – dvoznačna f-ja

1) $y = \sqrt{8x^2-x^4}$

2) $y = -\sqrt{8x^2-x^4}$

---

1) $y = \sqrt{-x^4+8x^2}$

1° $D(f)$: $-x^4+8x^2 \geq 0 \Rightarrow -x^2+8 \geq 0 \Rightarrow -2\sqrt{2} \leq x \leq 2\sqrt{2}$

2° Nule: $-x^4+8x^2=0 \Rightarrow -x^2(x^2-8)=0 \Rightarrow x_1=0$, $x_2=-2\sqrt{2}$, $x_3=2\sqrt{2}$

3° Simetrija: $f(-x) = \sqrt{-x^4+8x^2}$

što znači da je parno simetrična.

4° Izvodi:

$$(\sqrt{u})' = \frac{u'}{2\sqrt{u}} \Rightarrow \left(\sqrt{-x^4+8x^2}\right)' = \frac{-4x^3+8x}{2\sqrt{-x^4+8x^2}} = \frac{-2x^3+8x}{\sqrt{-x^4+8x^2}} = f'(x)$$

$$f''(x) = \frac{(-6x^2+8)\left(\sqrt{-x^4+8x^2}\right) - \left(\frac{-2x^3+8x}{\sqrt{-x^4+8x^2}}\right)\cdot(-2x^3+8x)}{-x^4+8x^2}$$

$$f''(x) = \frac{(-6x^2+8)\left(\sqrt{-x^4+8x^2}\right) - \frac{(-2x^3+8)^2}{\sqrt{-x^4+8x^2}}}{-x^4+8x^2} =$$

$$= \frac{(-6x^2+8)(-x^4+8x^2) - (4x^6-32x^3+64)}{(-x^4+8x^2)^{3/2}} =$$

$$= \frac{6x^6-14x^4-8x^4+64x^2-4x^6+32x^3+64}{(-x^4+8x^2)^{3/2}},$$

$$f''(x) = \frac{2x^6-22x^4+32x^3+64x^2+64}{(-x^4+8x^2)^{3/2}} = \frac{2x^3(x^2-16)-2x^2(11x-32)+64}{(-x^4+8x^2)^{3/2}}$$

$$f'(x) = \frac{-2x^3+8x}{\sqrt{-x^4+8x^2}} = 0, \quad x(-2x^2+8)=0$$

$x_1=0$, $x^2-4=0$

$x_2=-2$

$x_3=2$

5° Extremi:

Potrebno je ispitati tačke x_1, x_2 i x_3.

$x_1 = 0$

$f'(0)$ – nije definisano, a f-ja je neprekidna u $x=0$. Zbog toga je u $x=0$ moguć šiljak.

6° Tok f-je:

$$y' = \frac{-2x^3+8x}{\sqrt{-x^4+8x^2}} \Rightarrow \quad (-2\sqrt{2},-2)\ \nearrow \quad (-2,0)\ \swarrow \quad (0,2)\ \nearrow \quad (2,2\sqrt{2})\ \swarrow$$

Pomoću drugog izvoda ispitujemo konveksnost i konkavnost.

$(-2\sqrt{2},-2)$ interval $y''<0 \Rightarrow$ „$\cap$"

$(-2,0)$ $y''<0 \Rightarrow$ „$\cap$"

$(0,2)$ $y''<0 \Rightarrow$ „$\cap$"

$(2,2\sqrt{2})$ $y''<0 \Rightarrow$ „$\cap$"

7° Ugao ulaska u tačku $x=0$.

$$\lim_{x\to 0_-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0_-} \sqrt{\frac{8x^2-x^4}{x^2}} = \lim_{x\to 0_-} \sqrt{8-x^2} = \sqrt{8}$$

$\operatorname{tg}\alpha = \sqrt{8} \Rightarrow \alpha \approx 70°$

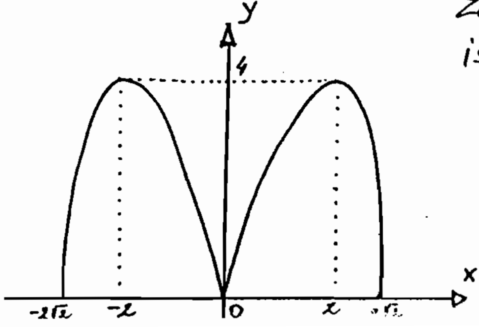
$$\lim_{x\to 0_+} \frac{f(x)-f(0)}{x-0} = \sqrt{8} \Rightarrow \alpha \approx 70°$$

8° Asimptote:

$$\lim_{x\to\pm\infty} \sqrt{-x^4+8x^2} = \sqrt{x^2(8-x^2)} = \text{nema } (\ominus \text{ ispod } \sqrt{\ })$$

$$\lim_{x\to\pm\infty} \frac{\sqrt{-x^4+8x^2}}{x} = \sqrt{-x^2+8} = \text{nema.}$$

9° Grafik



Za f-ju 2) je isto samo ispod x ose!

② Nacrtati f-ju : $f(x) = (3-x^2)e^x$.

1° $D(f)$: $\forall x \in R$

2° $f(-x) = (3-x^2)e^{-x} = \frac{(3-x^2)}{e^x} \neq f(x), \neq -f(x)$ pa nije simetrična

3° Nule: $3-x^2=0 \Rightarrow x^2=3 \Rightarrow x=\pm\sqrt{3}$

4° Izvodi:

$y' = e^x(-x^2-2x+3) \Rightarrow$ $D(f'(x))$: $\forall x \in R$

$y'=0$ za $(-x^2-2x+3)=0$

$\left.\begin{matrix} x_1 = -3 \\ x_2 = 1 \end{matrix}\right\}$ ST. Tačke, mogući ekstremi

$y'' = (e^x(-x^2-2x+3))' = e^x(-x^2-2x+3)+e^x(-2x-2) = e^x(-x^2-4x+1)$

$D(f'')$: $\forall x \in R$

$y''=0$ za $-x^2-4x+1=0 \Rightarrow$ $\left.\begin{matrix} x_1 = -2+\sqrt{5} \\ x_2 = -2-\sqrt{5} \end{matrix}\right\}$ mogući prevoj

5° Tok f-je:

F-ju podjelimo na intervale monotonosti:

$(-\infty,-3)$	$(-3,-\sqrt{3})$	$(-\sqrt{3},1)$	$(1,\sqrt{3})$	$(\sqrt{3},+\infty)$
↙	↗	↗	↙	↙

Ovdje su mete tačke kod kojih je moguće da f-ja iz rasta pređe u opadanje i obratno!

Prvim izvodom ispitajmo ove intervale:

$y'(-2) = e^{-2}(-(-2)^2-2(-2)+3) = \frac{1}{e^2}\cdot 3 > 0$ ↗

$y'(-4) = e^{-4}(-(-4)^2-2(-4)+3) < 0$ ↙

$y'(-1) = e^{-1}(-(-1)^2-2(-1)+3) > 0$ ↗

$y'(0,5) = e^{0,5}(-(0,5)^2-2(0,5)+3) < 0$ ↙

$y'(3) = e^3(-(3)^2-2(3)+3) < 0$ ↙

Sada znakove toka možemo upisati u gornju

6° Extremi:

Mogući kandidati su tačke $x_1 = -3$ i $x_2 = 1$.

$y''(-3) = e^{-3}(-(-3)^2 - 4(-3) + 1) = \frac{1}{e^3}(-9+12+1) = \frac{4}{e^3} > 0 \Rightarrow$ min.

$y''(1) = e^1(-(1)^2 - 4(1) + 1) = e(-1-4+1) = -4e < 0 \Rightarrow$ max.

Drugi izvodi su nam pokazali da su tačke -3 i 1 ekstremi i da je $x=-3$ tačka minimuma a $x=1$ tačka maximuma. Odredimo ordinate tih tačaka.

$y(-3) = (3-(-3)^2)e^{-3} = \frac{1}{e^3}(3-9) = -\frac{6}{e^3} \cong 0{,}298\ldots$

$y(1) = (3-1)e^1 = 2e \cong 5{,}436\ldots$

Dobili smo sljedeće ekstreme:

Ext. 1. $\left(-3, -\frac{6}{e^3}\right)$ /

Ext. 2. $(1, 2e)$ /

7° Prevojne tačke:

kandidati su: $x_1 = -2-\sqrt{5}$
$x_2 = -2+\sqrt{5}$.

Ako drugi izvod promjeni znak kada prođe kroz ove tačke znači da su to x-ordinate prevojne tačke. Zbog toga uzimamo test tačke između x_1 i x_2 i tražimo drugi izvod:

$y''(-5) = e^{-5}(-(-5)^2 - 4(-5) + 1) = \frac{1}{e^5}(-25+20+1) = -\frac{4}{e^5} < 0 \Rightarrow$ „$\cap$"

$y''(0) = 1 > 0 \Rightarrow$ „$\cup$"

$y''(1) = e(-(-1)^2 + (-4)(1) + 1) = e(-1-4+1) = -4e < 0 \Rightarrow$ „$\cap$"

Kako drugi izvod mijenja znak kroz ove tačke one su onda prevojne tačke.

$y(x_1) = (3-(-2-\sqrt{5})^2)e^{-2-\sqrt{5}} = \frac{(3-(4+4\sqrt{5}+5))}{e^{-(2+\sqrt{5})}} = \frac{-1-4\sqrt{5}-5}{e^{2+\sqrt{5}}} = \frac{-14{,}944}{69{,}135} = -0{,}216\ldots$

$y(x_2) = (3-(-2+\sqrt{5})^2)e^{-2+\sqrt{5}} = (3-(5-4\sqrt{5}+4))e^{-2+\sqrt{5}} = 2{,}944 \cdot 1{,}266 = 3{,}728\ldots$

Dobili smo dvije prevojne točke sa koordinatama:

Prev. 1. $(-2-\sqrt{5}, -0{,}261)$

Prev. 2. $(-2+\sqrt{5}, 3{,}728)$

8° Asimptote:

- Vertikalnih nema! (f-ja definisana $\forall x \in \mathbb{R}$)
- Horizontalne:

$$\lim_{x\to+\infty}(3-x^2)e^x = -\infty$$

$$\lim_{x\to-\infty}(3-x^2)e^x = \lim_{t\to+\infty}(3-t)\cdot\frac{1}{e^t} = \left|\begin{matrix}\text{nakon smjene}\\ x=-t,\ t\to+\infty\end{matrix}\right| =$$

$$= \lim_{t\to+\infty}\frac{3-t}{e^t} = 0.$$ $\boxed{y=0}$ je hor. ASM.

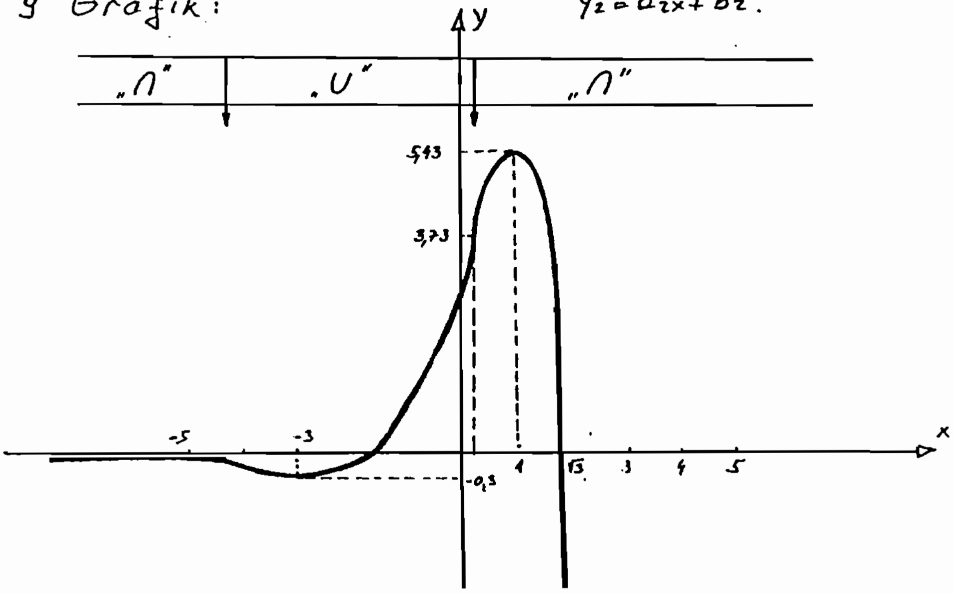
- kose asimptote:

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{(3-x^2)e^x}{x} = \lim_{x\to+\infty}\frac{\left(\frac{3}{x^2}-1\right)e^x}{\frac{1}{x}} = 0 = a_1$$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\left(\frac{3}{x^2}-1\right)e^x}{\frac{1}{x}} = 0 = a_2$$

Nemo kose asimptote! $y_1 = a_1x+b_1$

9° Grafik: $y_2 = a_2x+b_2$.



(Z) Ispitati tok i nacrtati graf f-je: $y=(x-a)e^{\frac{1}{x}}$, a - realan parametar.

1° $D(f)$: $\forall x \in \mathbb{R}/\{0\}$

kako f-ja nije definisana u tački $x=0$ tu je i prekidna. krajeve definisanosti $(0_+ ; 0_-)$ tj., prekidnosti ćemo odmah ispitati:

$$\lim_{x\to 0_-} f(x) = \lim_{x\to 0_-}(x-a)e^{1/x} = \lim_{\alpha\to 0}(-\alpha-a)e^{-1/\alpha} = \lim_{\alpha\to 0}\frac{-\alpha-a}{e^{1/\alpha}} = 0, \{V.AS.\}$$

$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+}(x-a)e^{1/x} = \lim_{\alpha\to 0}(\alpha-a)e^{1/\alpha} = \lim_{\alpha\to 0}\frac{e^{1/\alpha}}{\frac{1}{\alpha-a}} = \begin{cases} +\infty & za\ a\le 0 \\ -\infty & za\ a>0 \end{cases}$$

Imamo vertikalnu asimptotu $x=0$ kad $x\to 0_+$.

Osim ove tačke definicionog područja imamo i tačke $\pm\infty$.

$$\left.\begin{aligned} \lim_{x\to+\infty} f(x) &= \lim_{x\to+\infty}(x-a)e^{1/x} = +\infty \\ \lim_{x\to-\infty} f(x) &= \lim_{x\to-\infty}\frac{x-a}{e^0} = -\infty \end{aligned}\right\} \lim_{x\to\pm\infty} f(x) = \pm\infty \text{ - nema H.AS.}$$

2° Nule i znak f-je:

$f(x)=0$ za $x-a=0 \Rightarrow x=a$

$f(x)>0$ za $x>a \wedge x\neq 0$ jer je $e^{1/x}$ uvijek pozitivno, pa ne utiče na znak f-je.

$f(x)<0$ za $x<a \wedge x\neq 0$

3° Extremi i intervali monotonosti.

Najprije moramo naći izvode:

$$y' = \left((x-a)e^{1/x}\right)' = (x-a)'e^{1/x} + (x-a)(e^{1/x})' = e^{1/x} + (x-a)e^{1/x}\cdot\left(\frac{1}{x}\right)' =$$

$$= e^{1/x} + (x-a)e^{1/x}\cdot\left(\frac{0-1}{x^2}\right) = e^{1/x} - \frac{(x-a)e^{1/x}}{x^2} = \frac{e^{1/x}(x^2-x+a)}{x^2}$$

$$y'' = \left(\frac{e^{1/x}(x^2-x+a)}{x^2}\right)' = \frac{\left(e^{1/x}(x^2-x+a)\right)'^{*}\cdot x^2 - (x^2)'\left(e^{1/x}(x^2-x+a)\right)}{x^4} =$$

$$* \quad \left(e^{1/x}(x^2-x+a)\right)' = (e^{1/x})'(x^2-x+a) + (x^2-x+a)'\cdot e^{1/x} =$$

$$= e^{1/x}\cdot\left(-\frac{1}{x^2}\right)(x^2-x+a) + e^{1/x}(2x-1) = e^{1/x}\left(-1+\frac{1}{x}-\frac{a}{x^2}\right) + e^{1/x}(2x-1) =$$

$$= e^{1/x}\left(-1+\frac{1}{x}-\frac{a}{x^2}+2x-1\right) = e^{1/x}\left(-2+\frac{1}{x}-\frac{a}{x^2}+2x\right) = \left(\frac{-2x^2+x-a+2x^3}{x^2}\right)e^{1/x} =$$

$$= \frac{e^{1/x}(2x^3-2x^2+x-a)}{x^2}$$

$$= \frac{\frac{e^{1/x}(2x^3-2x^2+x-a)}{x^2}\cdot x^2 - 2x(e^{1/x}(x^2-x+a))}{x^4} =$$

$$= \frac{e^{1/x}\left[(2x^3-2x^2+x-a)-2x(x^2-x+a)\right]}{x^4} = \frac{e^{1/x}(2x^3-2x^2+x-a-2x^3+2x^2-2xa)}{x^4} =$$

$$= \frac{e^{1/x}(-2xa+x-a)}{x^4}$$

Primjetimo da je $a < \frac{1-\sqrt{1-4a}}{2}$ za $0<a<\frac{1}{4}$. Naime, iz $\left(a < \frac{1-\sqrt{1-4a}}{2},\ 0<a<\frac{1}{4}\right) \Rightarrow 2a-1<\sqrt{1-4a} \Rightarrow 1-4a < 1-4a+4a^2 \Rightarrow$ $0<4a^2 \Rightarrow a^2>0$ što je tačno zbog $a>0$ (dovoljno bi bilo $a\neq 0$).

4° Prevojne tačke, konkavnost i konveksnost.

$y'' = \frac{e^{1/x}(-2xa+x-a)}{x^4} \Rightarrow$ 1) $x\neq 0$

2) $y''=0 \Rightarrow -2xa+x-a=0 \wedge x\neq 0$

$x(1-2a)-a=0 \wedge x\neq 0$

$x=\frac{a}{1-2a}$ za $a\neq 0$ i $a\neq\frac{1}{2}$.

Grafik nema prevojnih tačaka za $a=0$ ili za $a=\frac{1}{2}$.

$f''(x)>0 \Rightarrow x-2ax-a>0 \wedge x\neq 0 \Rightarrow$

$x>\frac{a}{1-2a}$ za $\left\{(a\neq 0 \text{ i } a<\tfrac{1}{2})\wedge(x<\tfrac{a}{1-2a} \text{ za } a\neq 0 \wedge a>\tfrac{1}{2})\right\}$.

$f''(x)<0 \Rightarrow x-2ax-a<0 \wedge x\neq 0 \Rightarrow$

$\left\{(x<\frac{a}{1-2a},\ a\neq 0 \wedge a<\frac{1}{2})\wedge(x>\frac{a}{1-2a} \text{ za } a\neq 0\wedge a>\frac{1}{2})\right\}$

Dakle, grafik f-je $f(x)$ ima prevojnu tačku sa apcisom: $x=\frac{a}{1-2a}$ i za $a\neq 0$ i $a\neq\frac{1}{2}$

5° Asimptote:

Več znamo da je prava $x=0$ V.AS kada $x\to 0+$.
Pošto je $\lim_{x\to\pm\infty} f(x)=\pm\infty$, to kriva nema horizontalnih AS.

$f'(x) = \frac{e^{1/x}(x^2-x+a)}{x^2}$

1° $D(f'(x))$: $x\neq 0$.

2° $y'=0$ za $x^2-x+a=0 \Rightarrow x_{1,2}=\frac{1\pm\sqrt{1-4a}}{2}$

$a<\frac{1}{4}$ da bi izraz bio realan

[illegible] nije prevojna tačka, niti je ekstrem jer [illegible] f-ja ima prekid.

III $(0<a<\frac{1}{4},\ x<0) \Rightarrow f(x) < x-a+1$ $\forall$., kriva je ispod asimptote.

Za $x>0$, dovoljno je primjetiti da se prevojna tačka nalazi između tačke minimuma i tačke maximuma, i da su sve tri te tačke iznad asimptote. Nećemo tražiti tačku presjeka asimptote sa krivom, jer pri tom treba rješiti transcedentnu jednačinu. Da je tačno pretposljednja tvrdnja vidi se iz relacija:

$$\frac{1-\sqrt{1-4a}}{2} < \frac{a}{1-2a} < \frac{1+\sqrt{1-4a}}{2} \quad \text{jer:}$$

$$1-4a < (1-2a)^2 = 1-4a+4a^2 \Rightarrow 1-4a < (1-2a)\sqrt{1-4a} \Rightarrow$$

$$\Rightarrow \begin{cases} \dfrac{1-\sqrt{1-4a}}{2} < \dfrac{a}{1-2a}, \\ \dfrac{1+\sqrt{1-4a}}{2} > \dfrac{a}{1-2a} \end{cases}$$

IV $a=\frac{1}{4} \Rightarrow x_1 = x_2 = \frac{1}{2}$ – apcisa prevojne tačke.

$f(\frac{1}{2}) = (\frac{1}{2}-\frac{1}{4})e^{1/2} = \frac{\sqrt{e}}{4} < \frac{5}{4}$ – ordinata prevojne tačke.

Prema tome prevojna tačka je iznad asimptote.

Pogledajmo da li imako kose asimptote: $y = ax+b$.

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{(x-a)e^{1/x}}{x} = \lim_{x\to\pm\infty} \left(1-\frac{a}{x}\right)e^{1/x} = 1.$$

$$b = \lim_{x\to\pm\infty} [f(x)-ax] = \lim_{x\to\pm\infty} [(x-a)e^{1/x}-x] = \lim_{x\to\pm\infty} [x(e^{1/x}-1)-ae^{1/x}] =$$

$$= \lim_{x\to\pm\infty} \left[\frac{e^{1/x}-1}{\frac{1}{x}} - ae^{1/x}\right] = 1-a, \text{ jer je } \lim_{x\to\pm\infty} \frac{e^{1/x}-1}{\frac{1}{x}} =$$

$$= \lim_{x\to\pm\infty} \frac{-e^{1/x}\cdot\frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x\to\pm\infty} e^{1/x} = 1; \quad \lim_{x\to\pm\infty} ae^{1/x} = a\cdot 1 = a$$

Primjenuli smo Lopitalovo praviLo jer su uslovi:
(– jednoznačne i derivabilne f-je pri čemu se derivacija $\varphi(x)$ ne poništava – a to je oznaka za f-ju u nazivniku)

17. za njegovu primjenu ispunjeni.

Imamo tri slučaja:

I) $a > \frac{1}{4} \Rightarrow f'(x) > 0, \forall x \neq 0$ jer pod korjenom nije nula, nego imaginaran broj. Ovo znači da f-ja tada stalno roste i nema ekstrema jer nema ni stacionarnih točaka: $x_{1,2} = \frac{1 \pm i}{2}$ → neki

II) $a = \frac{1}{4} \Rightarrow f'(x) \geq 0, \forall x \neq 0$. S.t.: $x = \frac{1}{2}$ ali prvi izvod u njoj ne mijenja znak, pa tu nema ekstrema. Treba vidjeti da to nije prevojna tačka.

III) $a < \frac{1}{4} \Rightarrow f'(x) > 0$ za $x \in \left(-\infty, \frac{1-\sqrt{1-4a}}{2}\right) \cup \left(\frac{1+\sqrt{1-4a}}{2}, +\infty\right)$

$f'(x) < 0$ za $x \in \left(\frac{1-\sqrt{1-4a}}{2}, \frac{1+\sqrt{1-4a}}{2}\right)$, tj., f-ja ima maximum za $x = \frac{1-\sqrt{1-4a}}{2}$, a minimum za $x = \frac{1+\sqrt{1-4a}}{2}$, a maximum postoji za $a \neq 0$.

To se dobije ovako:

$f'(x) = \frac{e^{1/x}(x^2 - x + a)}{x^2} = 0$ kada je $(x^2 - x + a) = 0$, a to se dešava samo kada je $a < \frac{1}{4}$. U tom slučaju mi dobijemo dva realna korjena.

$x_1 = \frac{1-\sqrt{1-4a}}{2}$, $x_2 = \frac{1+\sqrt{1-4a}}{2}$, tada izraz pišemo u obliku:

$(x^2 - x + a) = (x - x_1)(x + x_1) > 0$ za $(-\infty, x_1) \cup (+\infty, x_2)$, a manje od o za (x_1, x_2). x_1 postoji kad je $\neq 0$, zato a mora biti $\neq 0$.

IV) $a = 0 \Rightarrow$ $f(x) = x \cdot e^{1/x}$, asimptota $y = x + 1$

$\left.\begin{array}{l} x > 0 \\ e^{1/x} > 1 + \frac{1}{x} \end{array}\right\} \Rightarrow f(x) = xe^{1/x} > x + 1$ za $x > 0$.

$\left.\begin{array}{l} e^{1/x} > 1 + \frac{1}{x}, x \neq 0 \\ x < 0 \end{array}\right\}$ $f(x) = xe^{1/x} < x + 1$

U ovom slučaju f-ja ima samo minimum za $x = 1$ i to $Y_{min} = e$ dok prevojne točke ne postoje (iako u $x = 0$ grafik mijenja konveksnost u konkavnost): za $x < 0$, $f''(x) < 0$ tj. kriva je konkavna, dok je za $x > 0$, $f''(x) = \frac{e^{1/x}}{x^3} > 0$.

Kosa asimptota ima jednačinu : $y = x + 1 - a$.
Pošto imamo parametar a, on utiče na različit položaj asimptote prema krivoj, pa trebamo pogledati taj položaj.

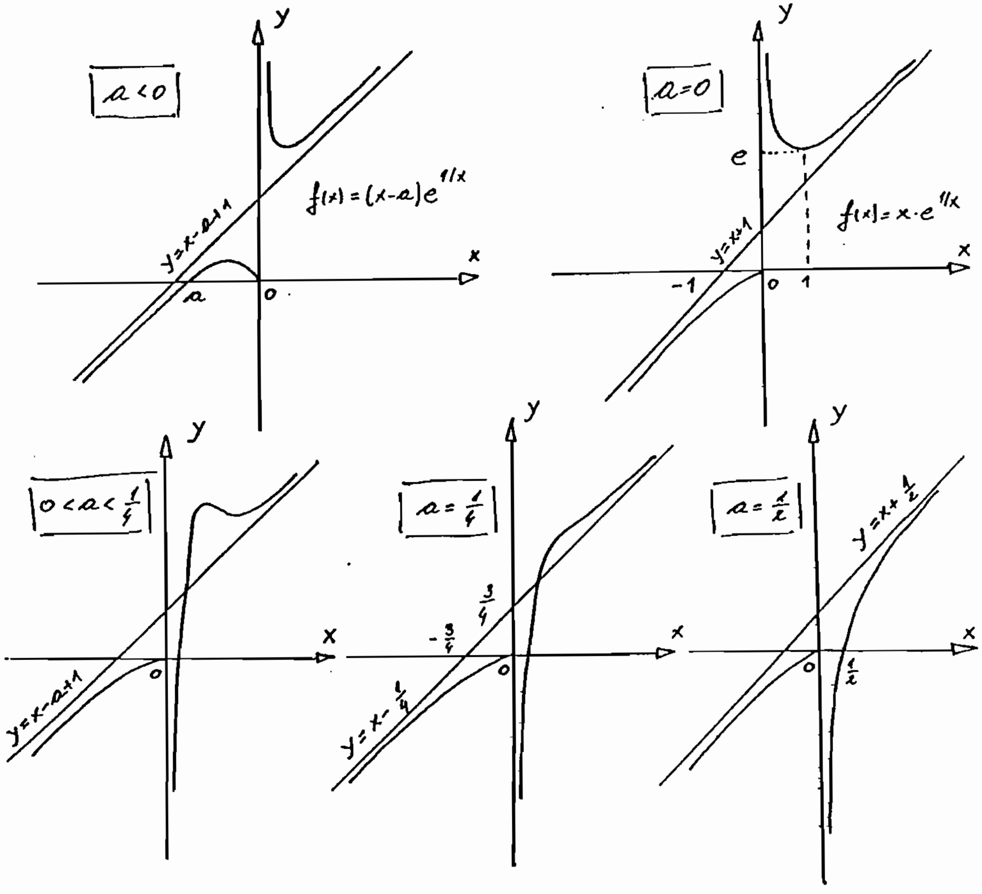
I) $\{a<0,\ x>0\ (x>a)\} \Rightarrow f(x) > x - a + 1$, tj., kriva je iznad asimptote za $x>0$ jer:

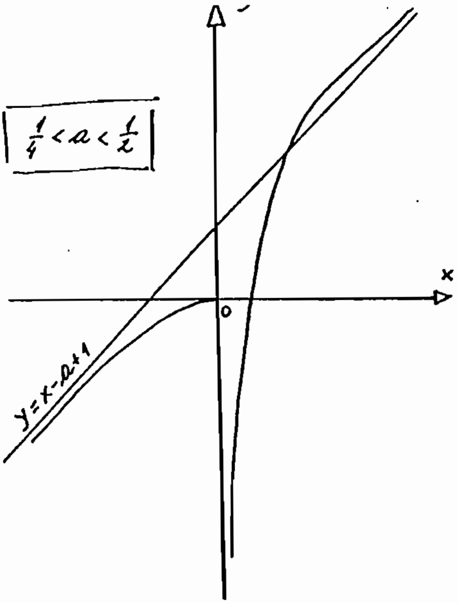
$$\left.\begin{array}{l} e^{1/x} > 1 + \frac{1}{x},\ x \neq 0 \\ \frac{1}{x} > \frac{1}{x-a},\ a<0<x \end{array}\right\} \Rightarrow e^{1/x} > 1 + \frac{1}{x-a} \Rightarrow$$

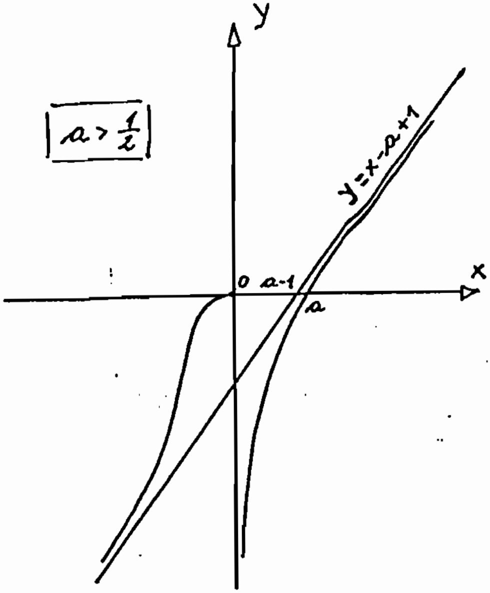
$$\Rightarrow f(x) = (x-a)e^{1/x} > (x-a)\left(1+\frac{1}{x-a}\right) = x - a + 1$$

$$\left.\begin{array}{l} (a<0,\ x<a) \Rightarrow \frac{1}{x} > \frac{1}{x-a} \\ e^{1/x} > 1 + \frac{1}{x},\ x \neq 0 \end{array}\right\} \Rightarrow f(x) = (x-a)e^{1/x} < x - a + 1$$

V) $a > \frac{1}{4} \Rightarrow \left\{ x = \frac{a}{1-2a} < 0 \text{ za } a > \frac{1}{2},\ x = \frac{a}{1-2a} > a \text{ za } \frac{1}{4} < a < \frac{1}{2} \right\}$







(2) Ispitati tok i nacrtati grafik f-je:

$y = f(x)^2$ ako je $f(x) = \operatorname{arctg} \frac{x+1}{x-1}$

RJ: Data f-ja ima oblik:

$g(x) = f(x^2) = \operatorname{arctg} \frac{x^2+1}{x^2-1}$

1° Definiciono područje:

$x^2 - 1 \neq 0 \Rightarrow x_{1,2} \neq \pm 1$ pa je: $D(f): (-\infty,-1) \cup (-1,1) \cup (1,+\infty)$

Imamo prekide u tačkama $x=-1$ i $x=1$.

2° Parnost i simetrija:

$g(-x) = \operatorname{arctg} \frac{x^2+1}{x^2-1} = g(x) \Rightarrow$ da je f-ja parno sim.

tj. simetrična u odnosu na y osu.

3° kodomen:

$R(f): \quad -\frac{\pi}{2} < \operatorname{arctg} \frac{x^2+1}{x^2-1} < \frac{\pi}{2}$

4° Znak:

$$\operatorname{arctg} \frac{x^2+1}{x^2-1} \begin{cases} <0, & -1<x<1 \\ >0, & x \in (-\infty,-1) \cup (1,+\infty) \end{cases}$$

5° Ispitajmo točku 1.

$\lim_{x\to 1-} \operatorname{arctg} \frac{x^2+1}{x^2-1} = -\frac{\pi}{2}$

$\lim_{x\to 1+} \operatorname{arctg} \frac{x^2+1}{x^2-1} = \frac{\pi}{2}$

6° Rast i opadanje:

$\operatorname{arctg} \frac{x^2+1}{x^2-1} = \operatorname{arctg} \frac{x^2-1+2}{x^2-1} = \operatorname{arctg}\left(1+\frac{2}{x^2-1}\right)$

max. za $x=0$ i to $y(max) = \operatorname{arctg}(-1) = -\frac{\pi}{4}$

za $x\in[0,1]$ f-ja strogo opada od $-\frac{\pi}{4}$ do $-\frac{\pi}{2}$;

a za $x\in(1,+\infty)$ f-ja strogo opada od $\frac{\pi}{2}$ do $\frac{\pi}{4}$;

$y=\frac{\pi}{4}$ je asimptota za $x\to\pm\infty$.

7° Granični položaj lijeve i desne tangente kad x raste prema 1 i kad x opada prema 1.

$$y' = \frac{1}{1+\left(\frac{x^2+1}{x^2-1}\right)^2}\cdot\frac{-4x}{(x^2-1)^2} = -\frac{1}{x^4+1}\cdot 2x \begin{cases} <0, & (x>0,\ x\neq 1) \\ =0, & (x=0) \\ >0, & (x<0,\ x\neq -1) \end{cases}$$

$\left.\begin{matrix}\lim_{x\to 1-} g'(x) = -1, & \lim_{x\to 1+} g'(x) = -1\end{matrix}\right\}$ uglovi ulaska u $x=1$.

Jednačine tangente: $y+\frac{\pi}{2} = -(x-1) \Leftrightarrow y = -x+1-\frac{\pi}{2}$

-granični položaj lijeve tangente ili kad $x\to 1-$; dok je granični položaj desne tangente $(x\to +1)$: $y-\frac{\pi}{2} = -(x-1) \Rightarrow$

$y = -x+\left(1+\frac{\pi}{2}\right)$

8° konkavnost i konveksnost:

$$g''(x) = \frac{-2(x^4+1)+8x^4}{(x^4+1)^2} = \frac{2(3x^4-1)}{(x^4+1)^2} = \frac{2(\sqrt{3}x^2-1)(x^2\sqrt{3}+1)}{(x^4+1)^2} = \frac{2\sqrt{3}\left(x-\frac{1}{\sqrt[4]{3}}\right)\left(x+\frac{1}{\sqrt[4]{3}}\right)(x^2\sqrt{3}+1)}{(x^4+1)^2}$$

$$g''(x) \begin{cases} >0, & x<-\frac{1}{\sqrt[4]{3}}, \text{ konkavna} \\ =0, & x=-\frac{1}{\sqrt[4]{3}}, \text{ prevojna točka} \\ <0, & -\frac{1}{\sqrt[4]{3}}<x<\frac{1}{\sqrt[4]{3}}, \text{ konveksna} \\ =0, & x=\frac{1}{\sqrt[4]{3}}, \text{ prevojna točka} \\ >0, & x>\frac{1}{\sqrt[4]{3}}, (x\neq 1) - \text{konkavna} \end{cases}$$

VAŽNO:

Primjetimo da je $\operatorname{arctg}\frac{x^2+1}{x^2-1} = \begin{cases} -\frac{\pi}{4} - \operatorname{arctg}x^2 & za\ -1<x<1 \\ \frac{3\pi}{4} - \operatorname{arctg}x^2 & za\ x\in(-\infty,-1)\cup(1,+\infty) \end{cases}$

* Stvarno, za $|x|<1$ imamo:

$$\operatorname{arctg}\frac{x^2+1}{x^2-1} = -\operatorname{arctg}\frac{1+x^2}{1-x^2} = -\operatorname{arctg}\frac{\operatorname{tg}\frac{\pi}{4}+\operatorname{tg}(\operatorname{arctg}x^2)}{1-\operatorname{tg}\frac{\pi}{4}\cdot\operatorname{tg}(\operatorname{arctg}x^2)} =$$

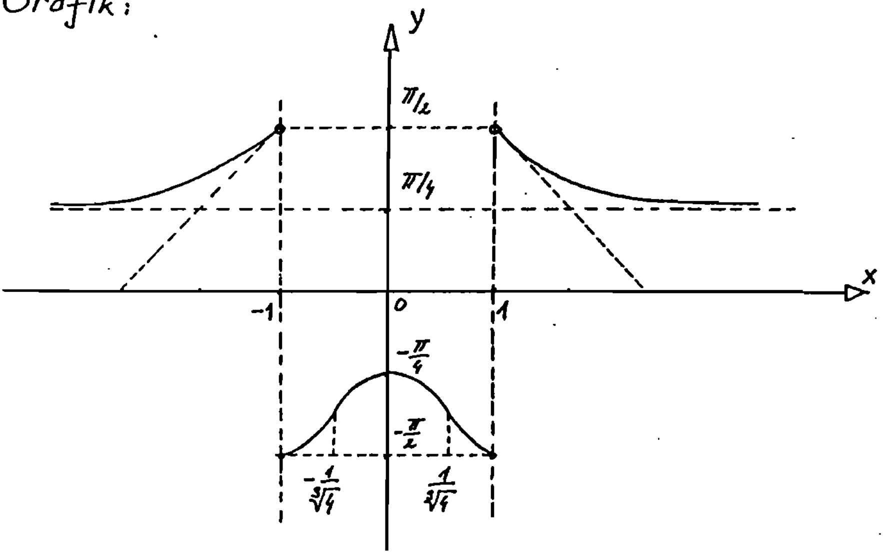
$= -\operatorname{arctg}\left[\operatorname{tg}\left(\frac{\pi}{4}+\operatorname{arctg}x^2\right)\right] = -\frac{\pi}{4} - \operatorname{arctg}x^2$, a za $|x|>1$, $(x=\frac{1}{y}, |y|<1) \Rightarrow$

$$\operatorname{arctg}\frac{x^2+1}{x^2-1} = \operatorname{arctg}\frac{\frac{1}{y^2}+1}{\frac{1}{y^2}-1} = \operatorname{arctg}\frac{1+y^2}{1-y^2} = \frac{\pi}{4}+\operatorname{arctg}y^2 = \frac{\pi}{4}+\operatorname{arctg}\frac{1}{x^2} =$$

$= \frac{\pi}{4}+\operatorname{arctg}\frac{1}{x^2} = \frac{3\pi}{4} - \operatorname{arctg}x^2$, jer $\operatorname{arctg}x + \operatorname{arctg}\frac{1}{x} = \frac{\pi}{2}\cdot\operatorname{sgn}x$,

$$\Rightarrow \operatorname{arctg}x^2 + \operatorname{arctg}\frac{1}{x^2} = \frac{\pi}{2}$$

9° Grafik:



② Nacrtati grafik f-je: $x^4 - 2x^2 = y^3(x-1)$

<u>RJ:</u> $y = \sqrt[3]{\frac{x^4-2x^2}{x-1}}$

1° $D(f)$: $x \neq 1$, treći korjen nema ograničenja

2° Nule f-je: $x^2(x^2-2)=0 \Rightarrow x_{1,2}=0,\ x_{3,4}=\pm\sqrt{2}$

3° Znak f-je:

$y = \sqrt[3]{\frac{x^4-2x^2}{x-1}} \lesseqgtr 0 \quad \Rightarrow$

$$y=\sqrt[3]{\frac{x^4-2x^2}{x-1}}=\begin{cases} <0, & -\infty<x<-\sqrt{2}\\ =0, & x=-\sqrt{2}\\ >0, & -\sqrt{2}<x<0\\ =0, & x=0\\ >0, & 0<x<1\\ \to+\infty, & x\to 1_-\\ \to-\infty, & x\to 1_+\\ <0, & 1<x<\sqrt{2}\\ =0, & x=\sqrt{2}\\ >0, & \sqrt{2}<x<+\infty \end{cases}$$

$f(0)=y_{min}=0$

$$a=\lim_{x\to\pm\infty}\frac{y}{x}=\lim_{x\to\pm\infty}\sqrt{\frac{x^2-2}{x(x+1)}}=1$$

$$b=\lim_{x\to\pm\infty}(y-ax)=\lim_{x\to\pm\infty}\left(\sqrt[3]{\frac{x^4-2x^2}{x-1}}-x\right)=\lim_{x\to\pm\infty}\frac{\sqrt[3]{x^4-2x^2}-\sqrt[3]{x^4-x^3}}{\sqrt[3]{x-1}}=$$

$$=\lim_{x\to\pm\infty}\frac{\sqrt[3]{x^2}\left(\sqrt[3]{x^2-2}-\sqrt[3]{x^2-x}\right)}{\sqrt[3]{x-1}}=\left|\sqrt[3]{a}-\sqrt[3]{b}=\frac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}\right|=$$

$$=\lim_{x\to\pm\infty}\frac{\sqrt[3]{x^2}\left[(x^2-2)-(x^2-x)\right]}{\sqrt[3]{x-1}}=\lim_{x\to\pm\infty}\frac{\sqrt[3]{x^2}\left[(x^2-2)-(x^2-x)\right]}{\sqrt[3]{x-1}\left[\sqrt[3]{(x^2-2)^2}+\sqrt[3]{(x^2-2)(x^2-x)}+\sqrt[3]{(x^2-x)^2}\right]}=\frac{1}{3}.$$

$y=ax+b \Rightarrow$ k.AS. $y=x+\frac{1}{3}\quad(x\to\pm\infty)$

Položaj krive prema asimptotama:

$$\sqrt[3]{\frac{x^4-2x^2}{x-1}}=x+\frac{1}{3}\,\Big/^3 \Rightarrow \frac{x^4-2x^2}{x-1}=x^3+x^2+\frac{1}{3}x+\frac{1}{27}\,\Big/\cdot(x-1)\Rightarrow$$

$$x^4-2x^2=x^4+x^3+\frac{x}{3}+\frac{x}{27}-(x^3+x^2)-\frac{1}{3}x-\frac{1}{27}\,\Big/\cdot 27$$

$$-54x^2=9x^2+x-27x^2-9x-1$$

$$-54x^2+27x^2-9x^2-x+9x+1=0 \Rightarrow -36x^2+8x+1=0\quad/(-1)\Rightarrow$$

$$\Rightarrow 36x^2-8x-1=0\qquad x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\Rightarrow x_{1,2}=\frac{8\pm\sqrt{64+144}}{72}=\frac{8\pm4\sqrt{13}}{72}$$

Dobili smo dva realna rješenja što znači da se kriva i asimptota sijeku!

Gdje je asimptota iznad, a gdje ispod krive?

Neka je: $\sqrt[3]{\frac{x^4-2x^2}{x-1}}<x+\frac{1}{3}\,\Big/^3\Rightarrow\frac{x^4-2x^2}{x-1}<x^3+x^2+\frac{x}{3}+\frac{1}{27}\,\Big/(x-1),\ x>1$

$\Rightarrow 36x^2-8x-1>0$, za $x>x_4$ kriva je ispod asimptote za $x>1$, dok za $x<1$ imamo $36x^2-x-1<0$, $x_5=\frac{2-\sqrt{13}}{8}<x<x_4$.

Ispitajmo tačku $x=\sqrt{2}$.

$$f'_+(\sqrt{2}) = \lim_{x\to\sqrt{2}+} \frac{y}{x-\sqrt{2}} = \lim_{x\to\sqrt{2}+} \sqrt[3]{\frac{x^2(x-\sqrt{2})(x+\sqrt{2})}{(x-1)(x-\sqrt{2})^3}} = +\infty$$

$$f'_-(\sqrt{2}) = \lim_{x\to\sqrt{2}-} \sqrt[3]{\frac{x^2(x+\sqrt{2})}{(x-1)(x-\sqrt{2})^2}} = +\infty$$

prava linija $x=\sqrt{2}$ je tangenta usmjereno prema gore.

Ispitajmo tačku $x=0$.

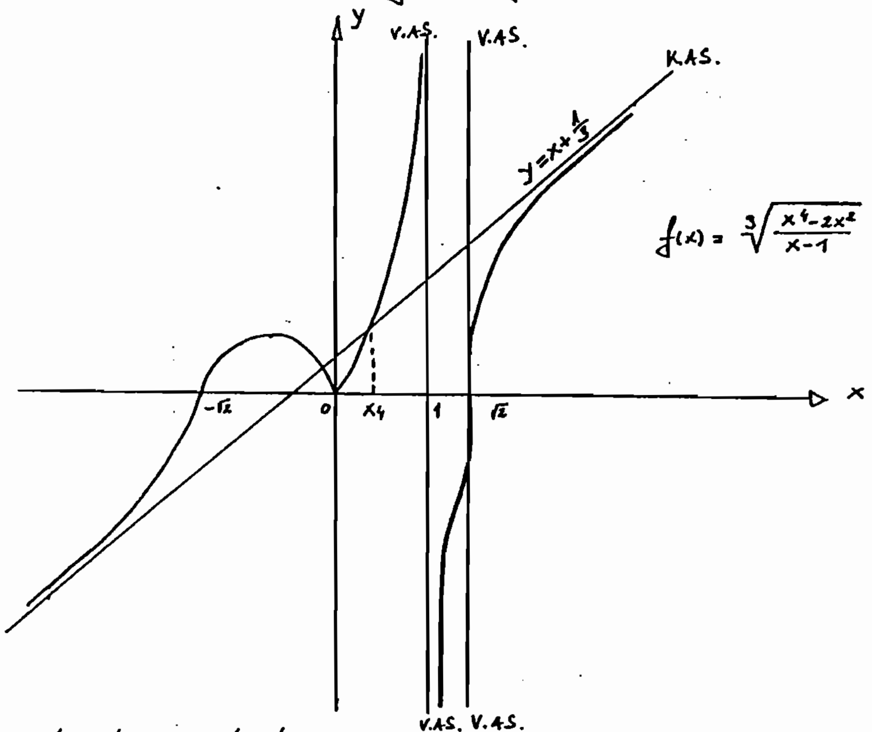
$$f'_+(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0+} \frac{y}{x} = \lim_{x\to 0+} \sqrt[3]{\frac{x^2-2}{x(x-1)}} = +\infty$$

$$f'_-(0) = \lim_{x\to 0-} \sqrt[3]{\frac{x^2-2}{x(x-1)}} = -\infty$$

$x=0$ je šiljak.

$$f'(x) = \frac{1}{3}\cdot\left(\frac{x^4-2x^2}{x-1}\right)^{-\frac{2}{3}} \cdot \frac{(4x^3-4x)(x-1)-x^4+2x^2}{(x-1)^2} = \frac{1}{3}\frac{1}{\sqrt[3]{x(x^2-2)}}\cdot\frac{3x^3-4x^2-2x+4}{(x-1)^{4/3}}$$

Sada možemo nacrtati grafik f-je:



Ispitajmo tačku $x=1$.

$$\lim_{x\to 1+} f(x) = \lim_{x\to 1+} \sqrt[3]{\frac{x^4-2x^2}{x-1}} = \sqrt[3]{\frac{1-2}{0+}} = \sqrt[3]{-\infty} = -\infty.$$

$$\lim_{x\to 1-} f(x) = \lim_{x\to 1-} \sqrt[3]{\frac{x^4-2x^2}{x-1}} = \sqrt[3]{\frac{1-2}{0-}} = \sqrt[3]{+\infty} = +\infty.$$

$x=1$ je V.AS.

(2) Ispitati precizno tok f-je : $f(x) = x^3 - 6x^2 + 9x - 4$ i nacrtati njen grafik :

RJ:

* $y = x^3 - 6x^2 + 9x - 4$ – polinom trećeg stepena ⇒

Opšta jednačina je : $a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = 0$ (1)
$a_n \neq 0$. koeficijenti su cijeli brojevi.

Ako je racionalan broj p/q, gdje su p i q relativno prosti i $p \cdot q \neq 0$ rješenja jednačine (1), tada je koeficijent a_n djeljiv sa q, a koeficijent a_0 djeljiv sa p.

Svako cjelobrojno rješenje jednačine (1) je djelilac koeficijenta a_0.

Ako je $a_n = 1$ tada je svako racionalno rješenje jednačine cio broj.

Šta ovo znači?

Kada imamo polinom nekog stepena tj. jednačinu kojoj tražimo nule : $a_n x^n + a_{n-1}x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0 = 0$ (sa cijelim koeficijentima), tada su moguća rješenja ovog polinoma brojevi p/q ($p \cdot q \neq 0$, relativno prosti), gdje je a_n djeljiv sa q, a a_0 djeljiv sa p.

Kod našeg polinoma : $y = x^3 - 6x^2 + 9x - 4$ tj. kod jednačine $y = x^3 - 6x^2 + 9x - 4 = 0$ imamo da je $a_n = 1$, to znači da će racionalna rješenja jednačine biti cijeli brojevi, i da će biti faktori koeficijenta $a_0 = -4$.

Primjer: $2x^3 - 9x + 4 = 0$ – Moguća rješenja su faktori od a_n i a_0 tj. p i q i p/q i $p \cdot q \neq 0$ i p i q – relativno prosti, tj. moguća rješenja su:
$\pm 1, \pm 2, \pm 4, \pm \frac{1}{2}, \pm \frac{2}{2} = 1, \pm \frac{4}{2} = 2$.

1° $D(f)$: $\forall x \in \mathbb{R}$

2° Simetrija: $f(-x) = -x^3 - 6x^2 - 9x - 4 \neq f(x) \neq -f(x)$ pa f-ja nije simetrična tj. nije ni parna ni neparna.

3° Nule f-je: $y = x^3 - 6x^2 + 9x - 4$

* primjenimo li pravilo o polinomima ova jednačina bi mogla imati nule:
$\pm 1, \pm 2, \pm 4$

$y=0$ za $x^3-6x^2+9x-4=0 \Rightarrow$

$\boxed{x=1}$	$1-6+9-4=0$	(T)
$x=2$	$8-24+18-4=-2$	(⊥)
$\boxed{x=4}$	$64-96+36-4=0$	(T)
$x=-1$	$-1-6-9-4<0$	(⊥)
$x=-2$	$-8-24-18-4<0$	(⊥)
$x=-4$	$-64-96-36-4<0$	(⊥)

našli smo dvije racionalne nule:

$(x^3-6x^2+9x-4):(x-1)=x^2-5x+4$

$\pm x^3 \mp x^2$

$-5x^2+9x-4$

$\mp 5x^2 \pm 5x$

$+4x-4$

$\pm 4x \mp 4$

$/\quad /$

$(x^2-5x+4):(x-4)=(x-1)$

$\pm x^2 \mp 4x$

$-x+4$

$-x+4$

$/\quad /$

Dobili smo da je $x=1$ dvostruka nula i da je $x=4$ treća nula.

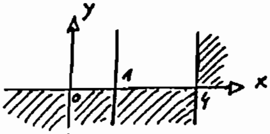
4° Znak f-je: Uzmimo test tačke koje se nalaze između nula i odredimo znak f-je:

TT: $x=-4 \Rightarrow y<0$

TT: $x=2 \Rightarrow y<0$

TT: $x=5 \Rightarrow y=16>0$

$\Rightarrow$ SKICA:



5° $y=x^3-6x^2+9x-4$

$y'=(x^3-6x^2+9x-4)'=3x^2-12x+9$, a) $D(f')$: $\forall x\in\mathbb{R}$

b) Nule su moguće tačke ekstrema:

$3x^2-12x+9=0\ /:3 \Rightarrow x^2-4x+3=0 \Rightarrow x_{1,2}=\frac{4\pm\sqrt{16-12}}{2}=\frac{4\pm2}{2}$

$\boxed{x_{1,2}=1,3}$ ST. I vrste.

Provjerimo da li su $x_1=1$ i $x_2=3$ ekstremi f-je:

$y''(1)=-6<0 \Rightarrow$ maximum

$y''(3)=6>0 \Rightarrow$ minimum.

$y''=(3x^2-12x+9)'=6x-12$, a) $D(f'')$: $\forall x\in\mathbb{R}$.

b) Nule su moguće prevojne tačke:

$6x-12=0\ /:6 \Rightarrow x-2=0 \Rightarrow \boxed{x=2}$ ST. II vrste.

Provjerimo da li je $x=2$ prevojna točka:

$y''(1)=-6<0 \Rightarrow$ „∩" konkavna

$y''(3)=6>0 \Rightarrow$ „∪" konveksna

Dobili smo:

ekstreme: $x_1=1 \Rightarrow y=1-6+9-4=0 \Rightarrow$ <u>Ext.1. (1,0) max.</u>

$x_2=3 \Rightarrow y=27-54+27-4=-4 \Rightarrow$ Ext.2. (3,-4) min.

prevoj: $x_1=2 \Rightarrow y=8-24+18-4=-2$ f-j, Prev. 1. (2,-2).

6° Intervali monotonosti koji se formiraju stacionarnim tačkama, a ja sam dodao i tačku $x=4$.

$(-\infty,1)$, $(1,3)$, $(3,4)$, $(4,+\infty)$

↗ ↘ ↗ ↗

$y'(-1) = 3+12+9 > 0$ ↗ rastuća

$y'(0) = 9 > 0$ ↗ rastuća

$y'(2) = 12-24+9 < 0$ ↘ opadajuća

$y'(3,5) = 36,75-42+9 = 3,75 > 0$ ↗ rastuća

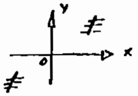
$y'(5) = 75-60+9 = 24 > 0$ ↗ rastuća

7° Ispitajmo još neke tačke definicionog područja:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(x^3-6x^2+9x-4) = \lim_{x\to+\infty}\left(x(x^2-6x+9)-4\right) = \underline{+\infty}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x^3\left(1-\frac{6}{x}+\frac{9}{x^2}-\frac{4}{x^3}\right) = \underline{-\infty}.$$

SKICA:



8° Asimptote:

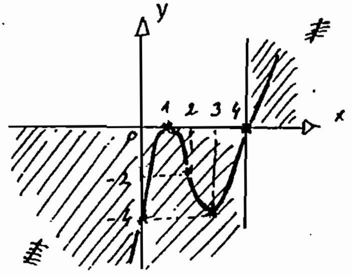
$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{x^3-6x^2+9x-4}{x} = \lim_{x\to+\infty}\left(\frac{1-\frac{6}{x}+\frac{9}{x^2}-\frac{4}{x^3}}{\frac{1}{x^2}}\right) = \frac{1}{0_+} = \underline{+\infty}.$$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \frac{1}{0_+} = \underline{+\infty}.$$

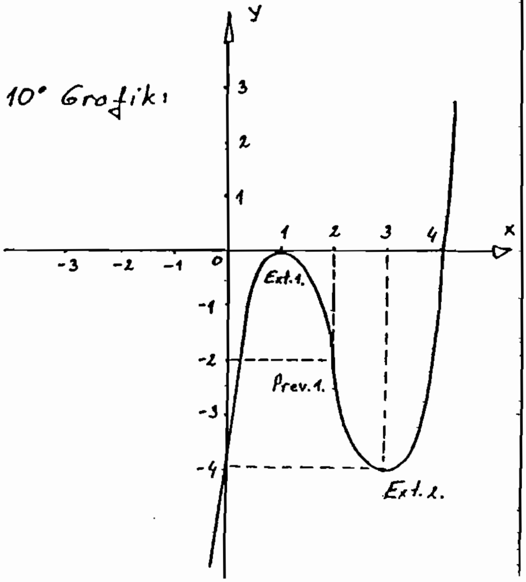
Nema kosu, horizontalnu ni vertikalnu asimptotu.

Za $x=0 \Rightarrow y=-4$

9° Skica:



10° Grafik:



② Nacrtati grafik f-je: $y=\dfrac{x^3-x^2-2x+2}{x}$.

RJ: $y=\dfrac{x^3-x^2-2x+2}{x}$

1° $D(f)$: $x\neq 0$.

2° Simetrija: $f(-x)=\dfrac{-x^3-x^2+2x+2}{-x}=\dfrac{x^3+x^2-2x-2}{x}\neq f(x),\neq -f(x)$.

F-ja nije simetrična.

3° Nule: $y=0$ za $x^3-x^2-2x+2=0$

* Prema teoremi o polinomima nule mogu biti neki od brojeva: $\pm 1, \pm 2$.

$x=1 \Rightarrow 1-1-2+2=0 \Rightarrow \boxed{x=1}$ je nula

$x=2 \Rightarrow 8-4-4+2=2$

$x=-1 \Rightarrow -1-1+2+2=2$

$x=-2 \Rightarrow -8-4+4+2=-6$

} Nisu nule polinoma.

Tražimo iracionalne nule:

$(x^3-x^2-2x+2):(x-1)=x^2-2$

$\underline{x^3-x^2}$ ($-\ +$)

$-2x+2$

$\underline{-2x+2}$ ($+\ -$)

$//\ /$

$(x^2-2)=(x-\sqrt{2})(x+\sqrt{2}) \Rightarrow$

$\underline{x^3-x^2-2x+2=(x-1)(x-\sqrt{2})(x+\sqrt{2})}$

Tražene nule su: $x_1=1,\ x_2=\sqrt{2},\ x_3=-\sqrt{2}$

4° Izvodi:

$\left(\dfrac{u}{v}\right)'=\dfrac{u'v-v'u}{v^2}$; $y'=\left(\dfrac{x^3-x^2-2x+2}{x}\right)'=\dfrac{2x^3-x^2-2}{x^2}$

a) $D(f')$: $x\neq 0$

b) Nule nam daju ST. I vrste.

$2x^3-x^2-2=0 \Rightarrow$ Moguće nule su: $\pm\frac{1}{2}, \pm 1, \pm 2$.

$x=-\frac{1}{2} \Rightarrow 2\left(-\frac{1}{2}\right)^3-\left(-\frac{1}{2}\right)^2-2=-2\neq 0$

$x=\frac{1}{2} \Rightarrow y\neq 0$

$x=-1 \Rightarrow y\neq 0$

$x=1 \Rightarrow y\neq 0$

$x=-2 \Rightarrow y\neq 0$

$x=2 \Rightarrow y\neq 0$

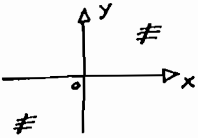
} NEMA RACIONALNIH NULA!

Provjerimo Limesom da li igdje ima nula!

$\lim\limits_{x\to-\infty}(2x^3-x^2-2)=-\infty$

$\lim\limits_{x\to+\infty}(x^2(2x-1)-2)=+\infty$

} $\Rightarrow$ SKICA:



Sa slikice vidimo da negdje mora imati nula jer f-ja $y=2x^3-x^2-2$ nema ni horizontalne ni vertikalne asimptote, a ni kose asimptote i mora negdje sjeći x osu.

To znači da mi imamo negdje stacionarnu točku koja može predstavljati minimum ili maximum.

$$y''=\left(\frac{2x^3-x^2-2}{x^2}\right)'=\frac{2x(x^3+2)}{x^4} \qquad a)\ D(f''):\ x\neq 0$$

b) $2x(x^3+2)=0$ za $x_1=0$, $x_2=\sqrt[3]{2}$ } ST. II vrste.

Razmotrimo moguće prevoje:

$x_1=0$ ne može nikako biti prevoj, jer f-ja nija definisana u toj točki!

$x_2=\sqrt[3]{2}$ – treba ispitati da li drugi izvod mjenja znak kada prolazi kroz ovu točku.

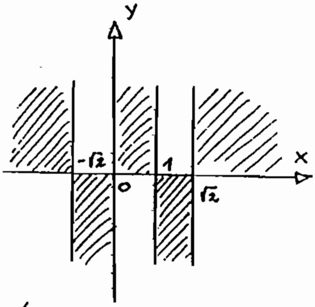
$$\left.\begin{aligned} &y''(-2)>0 \quad „\cup" \\ &y''(-1)<0 \quad „\cap" \\ &y''(1)>0 \quad „\cup" \\ &y''(\tfrac{1}{2})>0 \quad „\cup" \\ &y''(1{,}2)>0 \quad „\cup" \\ &y''(1{,}3)>0 \quad „\cup" \end{aligned}\right\} \ \sqrt[3]{2} \text{ – nije prevojna točka.}$$

5° Ispitajmo znak tok i definiciono područje f-je:

Uzmimo točke: $x=-\sqrt{2}$, $x=0$, $x=1$, $x=\sqrt{2}$

	$-\infty$ … $-\sqrt{2}$	$-\sqrt{2}$ … 0	0 … 1	1 … $\sqrt{2}$	$\sqrt{2}$ … $+\infty$
$x+\sqrt{2}$	–	+	+	+	+
x	–	–	+	+	+
$x-1$	–	–	–	+	+
$x-\sqrt{2}$	–	–	–	–	+
$f(x)$	+	–	+	–	+

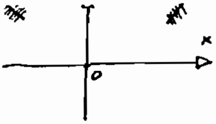
SKICA: ⇨



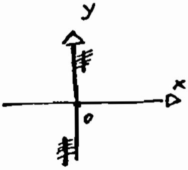
F-ja se nalazi u šrafiranim oblastima!

Limesom treba ispitati točke: $x=+\infty$, $x=-\infty$, $x=0$.

$$\begin{cases} \lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left(\dfrac{x^3-x^2-2x+2}{x}\right)=\lim\limits_{x\to+\infty}\left(\dfrac{1-\frac{1}{x}-\frac{2}{x^2}-\frac{2}{x^3}}{\frac{1}{x^2}}\right)=\underline{+\infty}. \\ \lim\limits_{x\to-\infty} f(x)=\underline{+\infty}. \end{cases}$$

⟹ SKICA: 

$$\left.\begin{array}{l}\lim\limits_{x\to 0_-} f(x) = \lim\limits_{x\to 0_-}\left(\dfrac{x^3-x^2-2x+2}{x}\right) = \dfrac{2}{0_-} = \underline{-\infty}.\\[2ex] \lim\limits_{x\to 0_+} f(x) = \dfrac{2}{0_+} = \underline{+\infty}.\end{array}\right\}$$

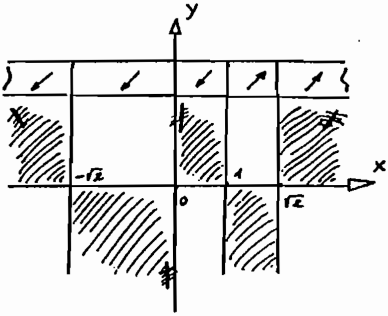
⟹ SKICA: 

V.AS. $x=0$

Za rost i opadanje f-je potrebno je uzeti test točke iz definicionog područja i ispitati ih prvim izvodom.

$$f'(x) = \frac{2x^3-x^2-2}{x^2}$$

$$\left.\begin{array}{lll} f'(2)>0 & \nearrow & \text{rastuća}\\ f'(1{,}2)>0 & \nearrow & \text{rastuća}\\ f'(\frac{1}{2})<0 & \swarrow & \text{opadajuća}\\ f'(-1)<0 & \swarrow & \text{opadajuća}\\ f'(-2)<0 & \swarrow & \text{opadajuća}\end{array}\right\}$$

SKICA: ⟹ 

6° Asimptote:

$$\left.\begin{array}{l}\lim\limits_{x\to+\infty}\dfrac{f(x)}{x} = \lim\limits_{x\to+\infty}\dfrac{x^3-x^2-2x+2}{x^2} = \dfrac{1}{0_+} = \underline{+\infty}.\\[2ex] \lim\limits_{x\to-\infty}\dfrac{f(x)}{x} = \dfrac{1}{0_-} = \underline{-\infty}.\end{array}\right\}$$

$x=0$: V.AS.

Nema K.AS ni H.AS.

Da li imamo krivolinijsku asimptotu?

$$\lim_{x\to+\infty}\frac{f(x)}{x^2} = \lim_{x\to+\infty}\frac{x^3-x^2-2x+2}{x^3} = \lim_{x\to+\infty}\left(\frac{1-\frac{1}{x}-\frac{2}{x^2}+\frac{2}{x^3}}{1}\right) = 1$$

Imamo, i koeficijent uz x^2 je 1.

To je kvadratna jednačina gdje je (ax^2+bx+c), $a=1$.

Pretpostavimo da znamo tačnu krivolinijsku asimptotu.

Tada će limes $(x\to+\infty)$ razlike naše f-je i ove asimptote težiti nuli.

Tada možemo pisati: $\lim\limits_{x\to+\infty}\left(f(x)-ax^2-bx-c\right)=0$ ⟹

$$\lim_{x\to+\infty}\left(\frac{x^3-x^2-2x+2}{x}-(ax^2+bx+c)\right)=0 \Rightarrow \lim_{x\to+\infty}\frac{(x^3-x^2-2x+2-ax^3-bx^2-cx)}{x}=$$

$$=\lim_{x\to+\infty}\left(\frac{x^3(1-a)-x^2(1+b)-x(2+c)}{x}+\frac{2}{x}\right)=$$

$$=\lim_{x\to+\infty}\left(\frac{x^3(1-a)-x^2(1+b)-x(2+c)}{x}\right)+\lim_{x\to+\infty}\frac{2}{x}=0$$

$$\lim_{x\to+\infty}\left(\frac{x^3(1-a)-x^2(1+b)-x(2+c)}{x}\right)+\lim_{x\to+\infty}\frac{2}{x}=0$$

Da bi ova jednakost bila zadovoljena mora biti:

$1-a=0 \Rightarrow a=1$
$1+b=0 \Rightarrow b=-1$
$2+c=0 \Rightarrow c=-2$

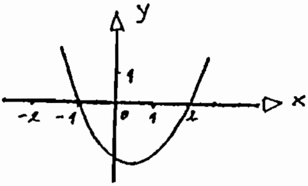
} to su koeficijenti naše krivolinijske asimptote.

$y=x^2-x-2$ – KRIV. AS. – Nacrtajmo je:

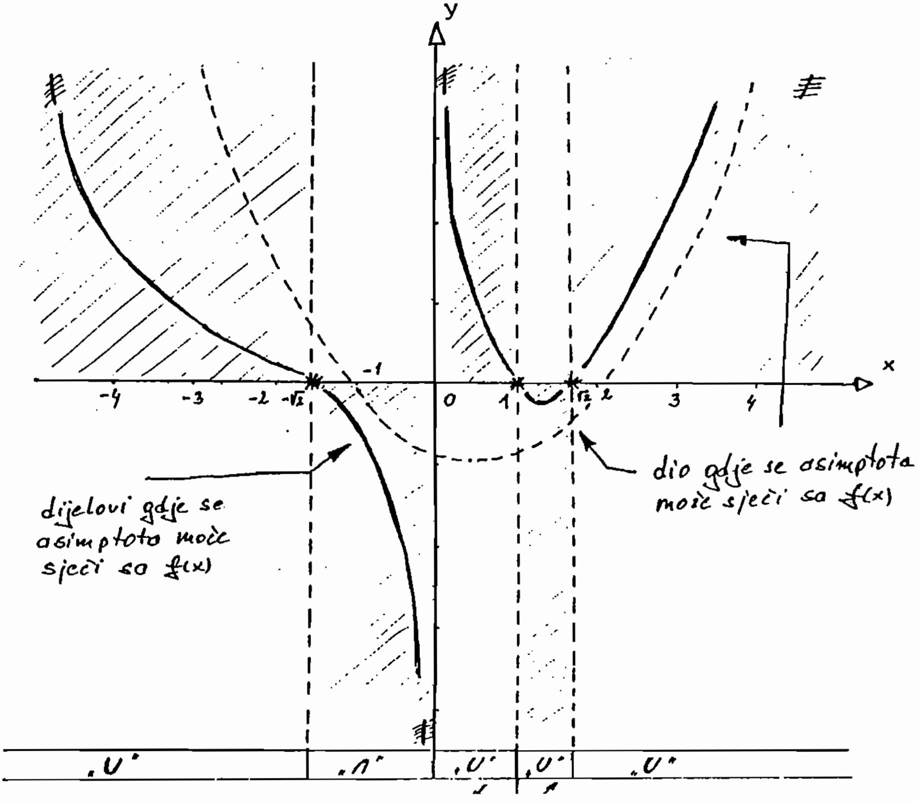
$y''=(x^2-x-2)''=(2x-1)'=2>0 \Rightarrow$ da je kriva „U" konveksna.

Nule: $x_{1,2}=\frac{1\pm\sqrt{1+8}}{2}=\frac{1\pm 3}{2} \Rightarrow \underline{x_1=-1}$, $\underline{x_2=2}$.

SKICA:
KRIV. AS.



7° Skica svega što smo našli:



8° Presjek asimptote sa krivom.

$$\frac{x^3-x^2-2x+2}{x^2} = x^2-x-2 \quad / \cdot x^2, \; x \neq 0$$

$$x^3-x^2-2x+2 = x^4-x^3-2x^2$$

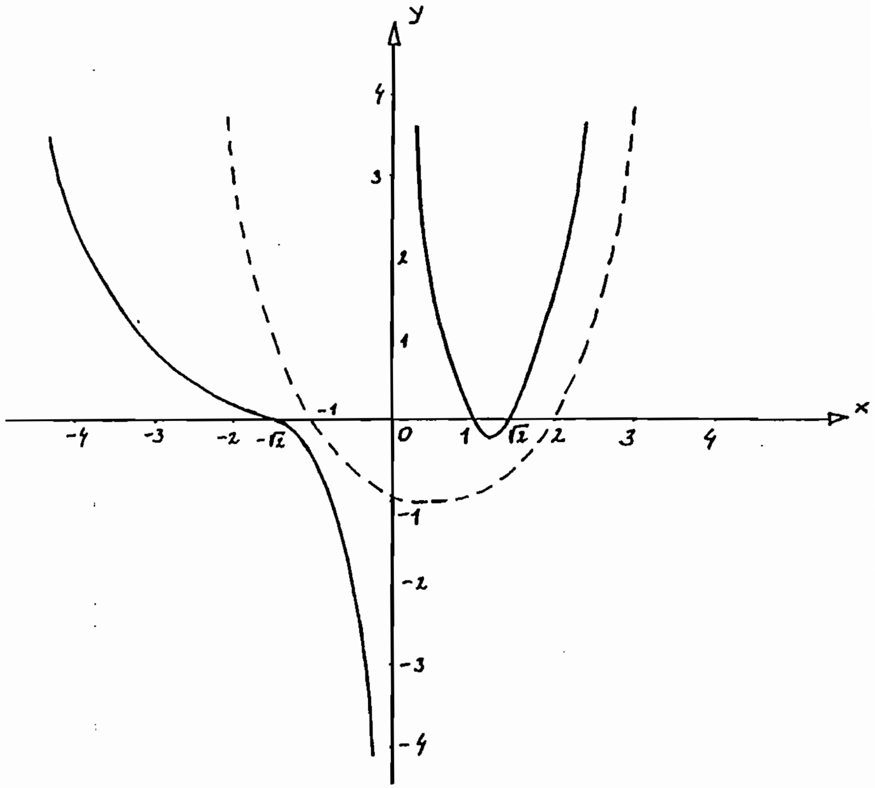
$$-x^4+x^3+x^3-x^2+2x^2-2x+2=0 \quad \Rightarrow \quad -x^4+2x^3+x^2-2x+2=0$$

Moguća rješenja su: $\pm 1, \pm 2$.

$x=1 \quad -1+2+2-2+2>0 \quad \perp$
$x=2 \quad -16+16+4-4+2>0 \quad \perp$
$x=-1 \quad -1-2+1+2+2>0 \quad \perp$
$x=-2 \quad -16-16+4+4+2<0 \quad \perp$

krlva se ne sječe sa asimptotom.

9° Grafik f-je:



EH

② Ispitati tok f-je i nacrtati krivu liniju $y=f(x)$, ako je : $f(x)=x^6-3x^4+3x^2$

RJ: $y=x^6-3x^4+3x^2$

1° $D(f)$: $\forall x \in R$

2° Simetrija: $y(-x)=x^6-3x^4+3x^2=y(x)$ – Fja je simetrična u odnosu na y osu, tj. parna je.

3° Nule f-je:

$y=0$ za $x^6-3x^4+3x^2=0 \Rightarrow x^2(x^4-3x^2+3)=0 \Rightarrow x_{1,2}=0 \wedge$

$\wedge\ x^4-3x^2+3=0 \Rightarrow |x^2=t| \Rightarrow t^2-3t+3=0 \Rightarrow t_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$t_{1,2}=\frac{3\pm\sqrt{9-12}}{2}$ ima imaginarne nule.

kako je $x^2\geq 0\ \forall x\in R$, vidimo da je i $x^4-3x^2+3\geq 0\ \forall x\in R$ tj. da je x^4-3x^2+3 uvijek nenegativna, pa je i $f(x)\geq 0$ $\forall x \in R$.

4° Izvodi :

$y'(x)=(x^6-3x^4+3x^2)'=(6x^5-12x^3+6x)$ 1° $D(f')$: $\forall x\in R$.

$y''(x)=(6x^5-12x^3+6x)'=(30x^4-36x^2+6)$ 2° $D(f'')$: $\forall x\in R$.

ST. I vrste, moguće su nule od: $6x^5-12x^3+6x=0\ /:6 \Rightarrow$

$x^5-2x^3+x=0 \Rightarrow x(x^4-2x^2+1)=0 \Rightarrow x_1=0,\ x^4-2x^2+1=0$

$x^4-2x^2+1=0 \Rightarrow |x^2=t| \Rightarrow t^2-2t+1=0$

$t_{1,2}=\frac{2\pm\sqrt{4-4}}{2}=\frac{2}{2}=1 \Rightarrow x^2=1$

$x=\pm 1$

ST. I.: $x_1=0, x_2=-1, x_3=1$

To su moguči ekstremi.

$y''(x)=30x^4-36x^2+6$

ST. II. za $y''=0$ tj. za $30x^4-36x^2+6=0\ /:6 \Rightarrow$

$5x^4-6x^2+1=0 \Rightarrow |x^2=t| \Rightarrow 5t^2-6t+1=0$

$t_{1,2}=\frac{6\pm\sqrt{36-20}}{10}=\frac{6\pm 4}{10}$, $t_1=\frac{1}{5}$, $t_2=1$

$x_1^2=\frac{1}{5} \Rightarrow x_1=\pm\sqrt{\frac{1}{5}}$

$x_2^2=1 \Rightarrow x_2=\pm 1.$

Stacionarne točke druge vrste su : $x_1=-1,\ x_2=-\frac{1}{\sqrt5},\ x_3=\frac{1}{\sqrt5},$ $x_4=1.$

To su moguči prevoji.

5° Extremi:

ST. I vrste: $x_1=0$, $x_2=-1$, $x_3=1$

$y''_{(x)}=30x^4-36x^2+6 \Rightarrow$ za $x=0 \Rightarrow y''_{(0)}=6>0$ minimum.

za $x=-1 \Rightarrow y''(-1)=0 \Rightarrow$ da moramo naći izvod višegreda. Ako bude paran a različit od nule tada ćemo imati ekstrem.

$y'''=120x^3-72x$ za $x=-1 \Rightarrow y'''(-1)=-120+72<0$ – prvi od izvoda višeg reda je različit od nule ali neparan pa ekstrem u toj tački ne postoji.

za $x=1 \Rightarrow y''_{(1)}=30-36+6=0 \Rightarrow y'''_{(1)}=120-72>0$ pa ni ovo nije tačka ekstrema.

Imamo samo da je $x=0$ tačka minimuma:

$y(0)=0 \Rightarrow$ Ext. 1. min $(0,0)$.

6° Prevojne tačke:

ST. II. $x_1=-1$, $x_2=-\frac{1}{\sqrt{5}}$, $x_3=\frac{1}{\sqrt{5}}$, $x_4=1$.

$y''_{(x)}=30x^4-36x^2+6$

$y''(-2)=30(-2)^4-36(-2)^2+6=480-144+6>0 \Rightarrow$ „U" konveksna.

$x=-1$

$y''(-\frac{1}{2})=30(-\frac{1}{2})^4-36(-\frac{1}{2})^2+6=1{,}875-9+6<0 \Rightarrow$ „∩" konkavna.

$x=-\frac{1}{\sqrt{5}}$

$y''(0)=6>0 \Rightarrow$ „U" konveksna.

$x=\frac{1}{\sqrt{5}}$

$y''(\frac{1}{2})=1{,}875-9+6<0 \Rightarrow$ „∩" konkavna.

$x=1$

$y''_{(2)}=480-144+6>0 \Rightarrow$ „U" konveksna.

Koordinate prevojnih tačaka:

$y=x^6-3x^4+3x^2$

$x_1=-1 \Rightarrow y_1=1$ — Pr. 1. $(-1,1)$

$x_2=-\frac{1}{\sqrt{5}} \Rightarrow y_2=\frac{301}{625}$ — Pr. 2. $(-\frac{1}{\sqrt{5}}, \frac{301}{625})$

$x_3=\frac{1}{\sqrt{5}} \Rightarrow y_3=\frac{301}{625}$ — Pr. 3. $(\frac{1}{\sqrt{5}}, \frac{301}{625})$

$x_4=1 \Rightarrow y_4=1$ — Pr. 4. $(1,1)$

7° Intervali monotonosti:

Formiraju se od tačaka koje se nazivaju kritične tačke, a to su tačke u kojima je $f'(x)=0$ ili u kojima $f'(x)$ ne postoji, to su ustvari stacionarne tačke I vrste (ST.I).

$(-\infty,-1)$, $(-1,-\frac{1}{\sqrt{5}})$, $(-\frac{1}{\sqrt{5}},0)$, $(0,\frac{1}{\sqrt{5}})$, $(\frac{1}{\sqrt{5}},1)$, $(1,+\infty)$.

↙ ↙ ↙ ↗ ↗ ↗

$y'(-2) = -192+96-12<0$ – opadajuća ↙

$y'(-\frac{1}{2}) = -0{,}1875+1{,}5-3=-1{,}6875<0$ – opadajuća ↙

$y'(-0{,}2) = -1{,}10432<0$ – opadajuća ↙

$y'(0{,}2) = 1{,}10432>0$ – rastuća ↗

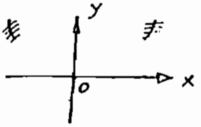
$y'(1) = 192-96+12>0$ – rastuća ↗

$y'(\frac{1}{2}) = 0{,}1875-1{,}5+3>0$ – rastuća ↗

8° Asimptote:

$y=x^6-3x^4+3x^2$

$$\begin{cases}\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x^2(x^4-3x^2+3) = \underline{+\infty}.\\ \lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x^2(x^4-3x^2+3) = \underline{+\infty}.\end{cases}$$



Nema V.AS.

$$\begin{cases}\lim\limits_{x\to+\infty} \frac{f(x)}{x} = \lim\limits_{x\to+\infty} \frac{x^6-3x^4+3x^2}{x} = \lim\limits_{x\to+\infty} \frac{1-\frac{3}{x^2}+\frac{3}{x^4}}{\frac{1}{x^5}} = \frac{1}{0_+} = \underline{+\infty}.\\ \lim\limits_{x\to-\infty} \frac{f(x)}{x} = \lim\limits_{x\to-\infty} \frac{1-\frac{3}{x^2}+\frac{3}{x^4}}{\frac{1}{x^5}} = \frac{1}{0_-} = \underline{-\infty}.\end{cases}$$

Nema K.AS niti H.AS.

9° Pogledajmo još ugao uloska u tačku nula:

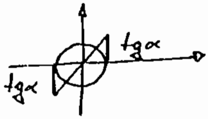
Znamo da je izvod koeficijent pravca na krivu, a to je ustvari tangens ugla tangente, tj. $f'(x)=tg\alpha$

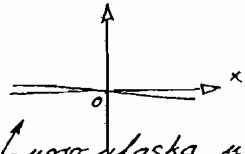
Ako pustimo da f-ja $f(x)$ teži ka 0_- vidjet ćemo gdje teži ugao tangente u toj tački.

$$\lim_{x\to0_-} \frac{f(x)-f(a)}{x-a} = \lim_{x\to0_-} \frac{f(x)-0}{x-0} = \lim_{x\to0_-} \frac{x^6-3x^4+3x^2}{x} =$$

$$= \lim_{x\to0_-} \frac{1-\frac{3}{x^2}+\frac{3}{x^4}}{\frac{1}{x^5}} = \frac{\infty}{\infty} = \lim_{x\to0_-} \frac{x^2(x^4-3x^2+3)}{x} = \lim_{x\to0_-} x(x^4-3x^2+3) = 0_-$$

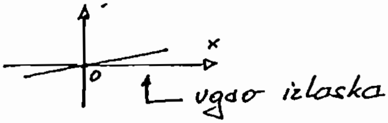
$tg\alpha = 0 \Rightarrow \alpha = 0°$ $\Rightarrow$





∠ ugao uloska u tačku nula.

$$\lim_{x \to 0_+} x(x^4 - 3x^2 + 3) = 0+ \quad \alpha = 0^\circ$$



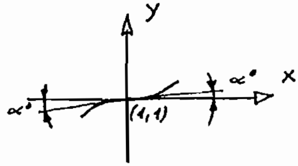
Treba imati na umu da je f-ja pozitivna i treba gledati konkavnost i konveksnost i tada se teško može pogriješiti!

Ispitajmo ugao ulaska i izlaska točke $x = 1$!

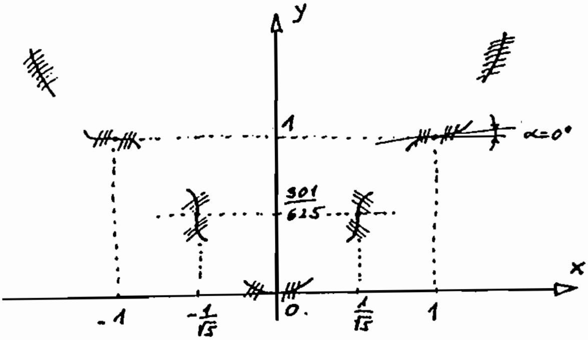
$$\lim_{x \to 1-} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1-} \frac{x^6 - 3x^4 + 3x^2 - 1}{x - 1} \overset{\frac{0}{0}}{=} \lim_{x \to 1-} \frac{0}{x - 1} = 0$$

$$tg\,\alpha = 0 \Rightarrow \quad \alpha_{ulaza} = 0^\circ$$

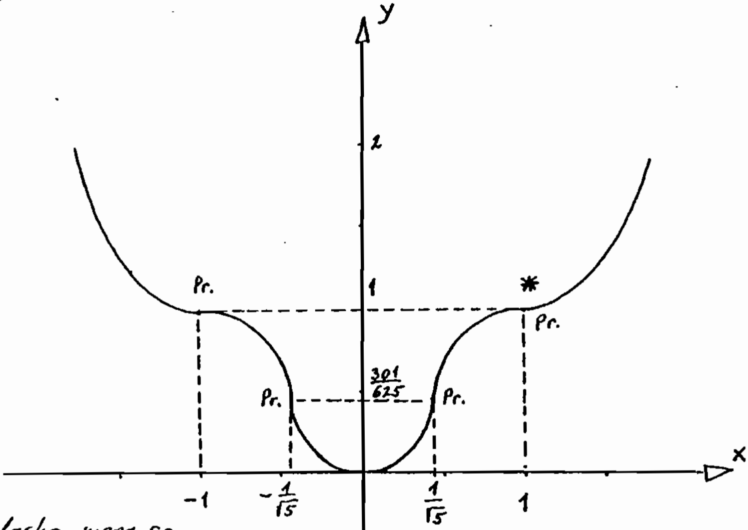
$$\lim_{x \to 1+} \frac{f(x) - f(a)}{x - a} = \lim_{x \to 1+} \frac{0}{x - 1} = 0 \Rightarrow \alpha = 0^\circ \ (izlaza)$$



10° Skica f-je:



11° Grafik:



* ugao ulaska mora se podudarati sa konveksnošću i konkavnošću f-je.

(2) Nacrtati grafik f-je $y=f(x)$ i ispitati njenu varijaciju, ako je: $f(x)=x^{2/3}(1-x)^{2/3}$

$$y=x^{2/3}(1-x)^{2/3}=\sqrt[3]{x^2(1-x)^2}$$

1° $D(f)$: $\forall x\in\mathbb{R}$ jer treći korjen nema ograničenja.

2° F-ja je pozitivna $\forall x\in\mathbb{R}$ tj. sav grafik f-je nalazi se iznad x ose osim eventualno na osi x.

3° Izvodi:

$$y'(x)=\left([x(1-x)]^{2/3}\right)'=\frac{2}{3}[x(1-x)]^{\frac{2}{3}-\frac{3}{3}}\cdot(x(1-x))'=\frac{2}{3}(x(1-x))^{-\frac{1}{3}}\cdot(x'(1-x)+x(1-x)')$$

$$=\frac{2}{3}\frac{1}{\sqrt[3]{x(1-x)}}\left((1-x)-x\right)=\frac{2}{3}\frac{(1-2x)}{\sqrt[3]{x(1-x)}}$$

1° $D(f'(x))$: treći korjen nema ograničenja ali mora biti.

$x(1-x)\neq 0 \Rightarrow \left.\begin{matrix}x_1\neq 0\\ x_2\neq 1\end{matrix}\right\}$ ST. I. vrste.

2° $f'(x)=0$ za $1-2x=0 \Rightarrow x=\frac{1}{2}\}$ ST. I. vrste.

$$y''(x)=\left[\frac{2}{3}\frac{(1-2x)}{\sqrt[3]{x(1-x)}}\right]'=\frac{2}{3}\left(\frac{(1-2x)}{\sqrt[3]{x(1-x)}}\right)'=\frac{2}{3}\frac{(1-2x)'\cdot\sqrt[3]{x(1-x)}-\left(\sqrt[3]{x(1-x)}\right)'(1-2x)}{\left(\sqrt[3]{x(1-x)}\right)^2}=$$

$$=\frac{2}{3}\frac{(-2)\sqrt[3]{x(1-x)}-\frac{(1-2x)^2}{3\cdot\sqrt[3]{x^2(1-x)^2}}}{\left(\sqrt[3]{x(1-x)}\right)^2}=\frac{(-6)\cdot\sqrt[3]{x(1-x)}\cdot\sqrt[3]{(x(1-x))^2}-(1-2x)^2}{3\cdot\sqrt[3]{(x(1-x))^2}\cdot\sqrt[3]{(x(1-x))^2}}\cdot\frac{2}{3}=$$

$$*\quad\left(\sqrt[3]{x(1-x)}\right)'=\left[(x(1-x))^{\frac{1}{3}}\right]'=\frac{1}{3}(x(1-x))^{\frac{1}{3}-\frac{3}{3}}\cdot(x(1-x))'=$$

$$=\frac{1}{3}(x(1-x))^{-\frac{2}{3}}\cdot(x(1-x))'=\frac{x'(1-x)+(1-x)'x}{3\cdot\sqrt[3]{(x(1-x))^2}}=\frac{1-2x}{3\cdot\sqrt[3]{(x(1-x))^2}}$$

$$=\frac{2}{3}\frac{(-6)(x(1-x))-(1-2x)^2}{3\cdot\sqrt[3]{(x(1-x))^4}}=\frac{2}{3}\frac{-6x+6x^2-1+4x-4x^2}{3\cdot\sqrt[3]{(x(1-x))^4}}=\frac{2}{9}\frac{2x^2-2x-1}{\sqrt[3]{(x(1-x))^4}}$$

1° $D(f''(x))$: - treći korjen nema ograničenja $\Rightarrow x\neq 0,\ x\neq 1$

2° $y''(x)=0$ za $2x^2-2x-1=0 \Rightarrow x_{1,2}=\frac{2\pm\sqrt{4+8}}{4}=\frac{2\pm\sqrt{4\cdot 3}}{4}=\frac{1\pm\sqrt{3}}{2}$

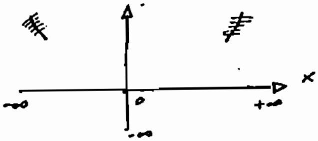
$\left.\begin{matrix}x_1=\frac{1+\sqrt{3}}{2}\cong 1{,}366\\ x_2=\frac{1-\sqrt{3}}{2}\cong -0{,}366\end{matrix}\right\}\Rightarrow$ ST. II. tj. mogući prevoji.

4° Ispitivanje definicionog područja:

$$\lim_{x\to+\infty}f(x)=\lim_{x\to+\infty}x^{2/3}\cdot(1-x)^{\frac{2}{3}}=\lim_{x\to+\infty}\sqrt[3]{(x(1-x))^2}=\lim_{x\to+\infty}\sqrt[3]{\left(x^2\left(\frac{1}{x}-1\right)\right)^2}=$$

$= \lim_{x\to+\infty} \sqrt[3]{x^4(-1)^2} = +\infty$

$\lim_{x\to-\infty} f(x) = +\infty$ } $\Longrightarrow$ SKICA:



6° Asimptote:

Kosa i horizontalna:

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\sqrt[3]{(x^2(\frac{1}{x}-1))^2}}{x} = \lim_{x\to+\infty} \sqrt[3]{\frac{(x^2(\frac{1}{x}-1))^2}{x^3}} = \lim_{x\to+\infty} \sqrt[3]{x(\tfrac{1}{x}-1)^2} =$$

$= \lim_{x\to+\infty} \sqrt[3]{x(\frac{1}{x}-1)^2} = +\infty$

$\lim_{x\to-\infty} \frac{f(x)}{x} = -\infty$ } Nema kose ni horizontalne asimptote!

7° Prema prvom izvodu možemo zaključiti kada je f-ja rastuća a kada opadajuća.

$y'(x) = \frac{2(1-2x)}{3\cdot\sqrt[3]{x(1-x)}}$ – koji faktori izraza utiču na predznak?

$1-2x=0 \Rightarrow x=\frac{1}{2}$

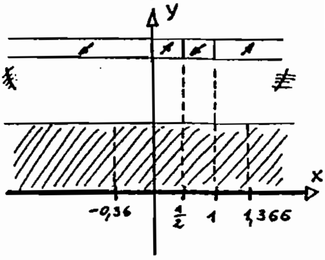
$x=0 \Rightarrow x=0$

$(1-x)=0 \Rightarrow x=1$ } $\Longrightarrow$

	$-\infty$ … 0	0 … $\frac{1}{2}$	$\frac{1}{2}$ … 1	1 … $+\infty$
x	−	+	+	+
1−2x	+	+	−	−
1−x	+	+	+	−
	−	+	−	+
TOK:	↘	↗	↘	↗

Ovo je ustvari tačka: „Intervali monotonosti".

* Sada smo dobili sljedeće:
  - $x=0$, $x=1$ – mogući šiljci
  - $x=0$, $x=1$ – moguće prevojne tačke
  - nema asimptota
  - nema izvoda u $x=0$ i $x=1$ a f-ja je tu definisano što asocira na šiljke.



Ostaje da provjerimo ekstreme, prevojne tačke i uglove ulaska i izlaska iz tih tačaka!

8° Extremi:

ST. I vrste su: $x_1=0$, $x_2=\frac{1}{2}$, $x_3=1$.

$y'(x) = \frac{2}{3}\frac{(1-2x)}{\sqrt[3]{x(1-x)}}$ $y'(0)$ – ne postoji, $y'(1)$ – ne postoji } Provjera za šiljak.

$y''(x) = \frac{2}{9}\frac{2x^2-2x-1}{\sqrt[3]{(x(x-1))^4}}$ $y''(0)$ – ne postoji, $y''(1)$ – ne postoji

$$y''\left(\tfrac{1}{2}\right) = \frac{2}{9}\cdot\frac{2\cdot\left(\frac{1}{2}\right)^2 - 2\cdot\frac{1}{2} - 1}{\underbrace{\sqrt[3]{\left(\frac{1}{2}\left(1-\frac{1}{2}\right)\right)^4}}_{>0}} < 0 \Rightarrow \text{u } x = \frac{1}{2} \text{ imamo maximum.}$$

$$y\left(\tfrac{1}{2}\right) = \sqrt[3]{\frac{1}{4}\cdot\frac{1}{4}} = \sqrt[3]{\frac{1}{16}} = 0{,}396.$$

9° Prevojne tačke:

ST. II.

$-\infty$ … $-0{,}366$	$-0{,}366$ … 0	0 … 1	1 … $1{,}366$	$1{,}366$ … ∞
$y''(-1)$	$y''(-0{,}1)$	$y''(\frac{1}{2})$	$y''(1{,}1)$	$y''(2)$
„U"	„∩"	„∩"	„∩"	„U"

$$y''(-1) = \frac{2}{9}\,\frac{2+2-1}{\sqrt[3]{(x(1-x))^4}} > 0 \Rightarrow \text{„U" konveksna}$$

$$y''(-0{,}1) = \frac{0{,}02+0{,}2-1}{>0} < 0 \Rightarrow \text{„∩" konkavna}$$

$$y''\left(\tfrac{1}{2}\right) < 0 \Rightarrow \text{„∩" konkavna}$$

$$y''(1{,}1) = \frac{2{,}24-2{,}2-1}{>0} < 0 \Rightarrow \text{„∩" konkavna}$$

$$y''(2) = \frac{2}{9}\cdot\frac{2+2-1}{>0} > 0 \Rightarrow \text{„U" konveksna}$$

Koordinate prevojnih tačaka:

$$y(-0{,}366) = \sqrt[3]{(0{,}366)^2(1+0{,}366)^2} \cong 0{,}629$$

$$y(1{,}366) = \sqrt[3]{(1{,}366)^2(1-1{,}366)^2} \cong 0{,}629$$

$\Rightarrow$ Prev. 1. $(-0{,}366,\ 0{,}629)$ Prev. 2. $(1{,}366,\ 0{,}629)$

Ispitajmo uglove uloska i izlaska u 0 i 1:

$$\lim_{x\to 0-}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 0-}\frac{\sqrt[3]{x^2(1-x)^2}}{x} = \lim_{x\to 0-}\sqrt[3]{\frac{(1-x)^2}{x}} = \frac{1}{0_-} = -\infty.$$

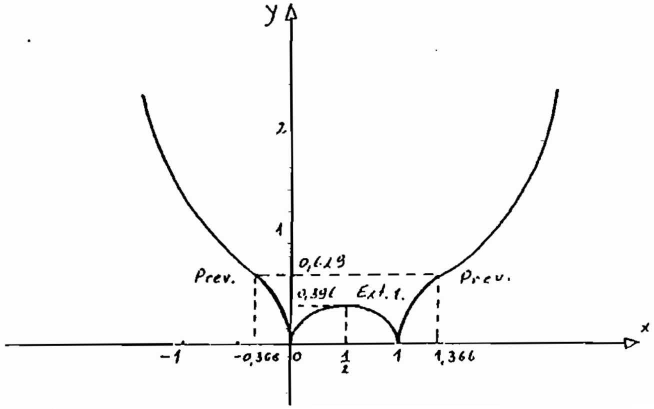
$$tg\,\alpha = -\infty \Rightarrow \alpha = 270°$$

$$\lim_{x\to 0+}\frac{f(x)-f(0)}{x-a} = \lim_{x\to 0+}\sqrt[3]{\frac{(1-x)^2}{x}} = \frac{1}{0_+} = +\infty \Rightarrow tg\,\alpha = +\infty \Rightarrow \alpha = 90°$$

$$\lim_{x\to 1-}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 1-}\frac{\sqrt[3]{x^2(1-x)^2}}{x-1} = \lim_{x\to 1-}\sqrt[3]{\frac{x^2}{x-1}} = -\infty. \Rightarrow tg\,\alpha = -\infty,\ \alpha = 270°$$

$$\lim_{x\to 1+}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 1+}\sqrt[3]{\frac{x^2}{x-1}} = +\infty. \Rightarrow tg\,\alpha = +\infty,\ \alpha = 90°$$

10° Grafik f-je:



② Ispitati tok f-je : $y=\frac{x^3}{3-x^2}$ i nacrtati njen grafik.

RJ: $y=\frac{x^3}{3-x^2}$

1° D(f): $3-x^2\neq 0 \Rightarrow -x^2\neq -3 \Rightarrow x^2\neq 3 \Rightarrow x\neq \pm\sqrt{3}$

Provjera: 1) $(x-\sqrt{3})(x+\sqrt{3})=x^2+x\sqrt{3}-x\sqrt{3}-3=\underline{x^2-3}$ ⊥

2) $(\sqrt{3}-x)(\sqrt{3}+x)=3+\sqrt{3}x-\sqrt{3}x-x^2=\underline{3-x^2}$ Ⓣ

2° Simetrija :

$y(-x)=-\frac{x^3}{3-x^2}=-y(x) \Rightarrow$ f-ja je neparno simetrična, tj., simetrična je u odnosu na koordinatni početak.

3° Tok f-je :

$y=\frac{x^3}{3-x^2}=\frac{x^3}{(\sqrt{3}-x)(\sqrt{3}+x)}$ TABLICA : $\Rightarrow$

	$-\infty$ … $-\sqrt{3}$	$-\sqrt{3}$ … 0	0 … $\sqrt{3}$	$\sqrt{3}$ … $+\infty$
$\sqrt{3}+x$	−	+	+	+
x	−	−	+	+
$\sqrt{3}-x$	+	+	+	−
	+	−	+	−

4° Nula f-je :

$x=0$. F-ja prolazi kroz koordinatni početak.

5° Izvodi :

$$y'_{(x)}=\left(\frac{x^3}{3-x^2}\right)'=\frac{(x^3)'(3-x^2)-(3-x^2)'\cdot x^3}{(3-x^2)^2}=\frac{3x^2(3-x^2)-x^3(-2x)}{(3-x^2)^2}=$$

$$=\frac{9x^2-3x^4+2x^4}{(3-x^2)^2}=\frac{-x^4+9x^2}{(3-x^2)^2}=\frac{x^2(9-x^2)}{(3-x^2)^2}=\underline{\frac{x^2(9-x^2)}{(3-x^2)^2}}.$$

$$y''_{(x)}=\left(\frac{9x^2-x^4}{(3-x^2)^2}\right)'=\frac{(9x^2-x^4)'(3-x^2)^2-((3-x^2)^2)'(9x^2-x^4)}{(3-x^2)^4}=$$

$$=\frac{(18x-4x^3)(3-x^2)^2-(2(3-x^2)^{2-1}(3-x^2)')(9x^2-x^4)}{(3-x^2)^4}=\frac{(18x-4x^3)(3-x^2)^2-(2(3-x^2)(-2x)(9x^2-x^4))}{(3-x^2)^4}=$$

$$=\frac{\cancel{(3-x^2)}((18x-4x^3)(3-x^2)-(2(-2x)(9x^2-x^4)))}{(3-x^2)^{\cancel{4}\,3}}=\frac{54x-18x^3-12x^3+4x^5+4x(9x^2-x^4)}{(3-x^2)^3}=$$

$$=\frac{54x-30x^3+4x^5+36x^3-4x^5}{(3-x^2)^3}=\underline{\frac{6x(x^2+9)}{(3-x^2)^3}}$$

$y'_{(x)}=\frac{x^2(9-x^2)}{(3-x^2)^2}$

1° $D(f'_{(x)}): x\neq\pm\sqrt{3}$

2° Nule: $y'(x)=0$ za $x=0$, $x=\pm 3$ } ST. I.

$y''_{(x)}=\frac{6x(x^2+9)}{(3-x^2)^3}$

1° $D(f''_{(x)}): x\neq\pm\sqrt{3}$

2° Nule: $x^2+9>0, \forall x\in R$; $x=0$ } ST. II.

6° Asimptote :

Primjetili smo da f-ja nije definisana za $x=\pm\sqrt{3}$ pa tu postoji mogućnost vertikalnih asimptota.

$$\lim_{x\to-\sqrt{3}(\uparrow)_-}\frac{x^3}{3-x^2}=\lim_{x\to-\sqrt{3}(\uparrow)_-}\frac{x^3}{\underbrace{(\sqrt{3}-x)}_{>0}\underbrace{(\sqrt{3}+x)}_{0_-}}=\underline{+\infty}$$

ili

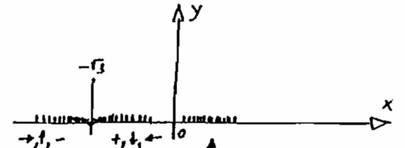
↳ gledamo x-eve koji su blizu $\sqrt{3}$.

$$\lim_{x\to-\sqrt{3}(\uparrow)_-}\frac{x^3}{3-x^2}=\lim_{x\to-\sqrt{3}(\uparrow)}\frac{x^3}{3-(-\sqrt{3})^2}=\frac{(-\sqrt{3})^3}{0_-}=\underline{+\infty}.$$

↳ po apsolutnoj vrijednosti brojevi manji od $-\sqrt{3}$ kada se kvadriraju su veći od 3 pa je razlika $3-(\text{broj}>3)$ manja od nule.

$$\lim_{x\to-\sqrt{3}\downarrow(+)}\frac{x^3}{3-x^2}=\lim_{x\to-\sqrt{3}\downarrow}\frac{x^3}{(\sqrt{3}-x)(\sqrt{3}+x)}=\lim_{x\to-\sqrt{3}+}\frac{-(\sqrt{3})^3}{\underbrace{(\sqrt{3}-x)}_{>0}\underbrace{(\sqrt{3}+x)}_{>0}}=\underline{-\infty}.$$

↳ gledamo brojeve koji su blizu $(-\sqrt{3})$ sa pozitivne str, desne strane.



ovi su bitni za $-\sqrt{3}$!

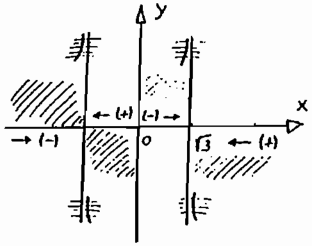
↑ ne trebamo gledati ove brojeve!

$$\lim_{x\to+\sqrt{3}\uparrow}\frac{x^3}{(\sqrt{3}-x)(\sqrt{3}+x)}=\frac{>0-\text{const}}{0_+}=\underline{+\infty}.$$

$$\lim_{x\to+\sqrt{3}\downarrow}\frac{x^3}{(\sqrt{3}-x)(\sqrt{3}+x)}=\lim_{x\to+\sqrt{3}\downarrow}\frac{x^3}{3-x^2\,]<0}=\underline{-\infty}.$$

↳ manje po aps vrijednosti od 3.

Dobili smo skicu:



Da li f-ja ima kosu asimptotu?

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x^3}{x(3-x^2)}=\lim_{x\to+\infty}\frac{x^3}{x(3-x^2)}=\lim_{x\to+\infty}\frac{x^3}{x^3\left(\frac{3}{x^2}-1\right)}=$$

$$=\lim_{x\to+\infty}\frac{1}{0-1}=\underline{-1}.$$

$$\lim_{x\to-\infty}\frac{1}{\left(\frac{3}{x^2}-1\right)}=\frac{1}{0-1}=\underline{-1}.$$

} koeficijent a uz x je -1.

$y=ax+b$ – KAS.

Potrošimo sada b!

$$\lim_{x\to+\infty}(f(x)-ax)=\lim_{x\to+\infty}\left(\frac{x^3}{3-x^2}+x\right)=\lim_{x\to+\infty}\left(\frac{x^3+x(3-x^2)}{(3-x^2)}\right)=\lim_{x\to+\infty}\left(\frac{x^3+3x-x^3}{(3-x^2)}\right)=$$

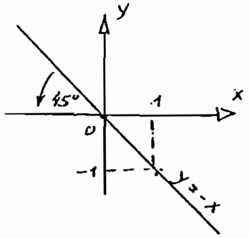
$$=\lim_{x\to+\infty}\frac{3x}{3-x^2}=\lim_{x\to+\infty}\frac{3/x}{3/x^2-1}=\frac{0_+}{-1}=\underline{0.}$$

$$\lim_{x\to-\infty}\frac{3x}{3-x^2}=\frac{0_-}{-1}=\underline{0.}$$

$\Big\}$ $b=0$

k.A.S. $y=-x$

SKICA:



Nemamo horizontalne asimptote. Ona se nađe tako što se traži b, u slučaju kada je $a=0$.

7° Extremi:

$$y'(x)=\frac{x^2(9-x^2)}{(3-x^2)^3}\Rightarrow$$

$x=\pm3$ – ST. I.

$x\neq\pm\sqrt3$ – sigurno nisu ekstremi jer f-ja nije definisana u ovim tačkama $x=\pm\sqrt3$ su takođe ST. I vrste.

$$y''(x)=\frac{6x(x^2+9)}{(3-x^2)^3}$$

$$y''(+3)=\frac{6\cdot3(3^2+9)}{(3-9)^3}\begin{matrix}>0\\<0\end{matrix}<0\Rightarrow \text{maximum}$$

$$y(3)=\frac{3^3}{3-9}=\frac{27}{-6}=-\frac{9}{2}$$

$\Big\}\Rightarrow$ **Ext. 1. $(3,-\frac{9}{2})$**

$$y''(-3)=\frac{6(-3)(9+9)}{(3-9)^3}\begin{matrix}<0\\<0\end{matrix}>0\Rightarrow \text{minimum}$$

$$y(-3)=\frac{-27}{-6}=\frac{9}{2}$$

$\Big\}\Rightarrow$ **Ext. 2. $(-3,\frac{9}{2})$**

8° Prevojne tačke konveksnost i konkavnost.

$$y''(x)=\frac{6x(x^2+9)}{(3-x^2)^3}$$

$x=0$ – ST. II

$x\neq\pm\sqrt3$ – ne mogu biti prevoji jer f-ja nije definisana u tim tačkama.

Napravimo tabelu:

Prev. 1. $(0,0)$

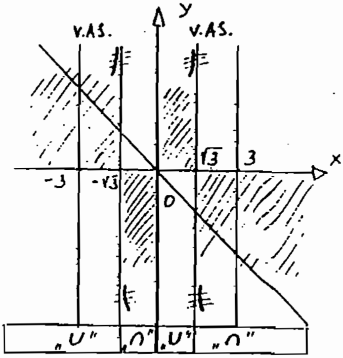
	$-\infty$ … $-\sqrt3$	$-\sqrt3$ … 0	0 … $\sqrt3$	$\sqrt3$ … $+\infty$
$\sqrt3+x$	−	0 +	+	+
x	−	−	0 +	+
$\sqrt3-x$	+	+	+	0 −
	+	−	+	−
$y''(x)$	>0	<0	>0	<0
	„U"	„∩"	„U"	„∩"

9° Pogledajmo još tok f-je:

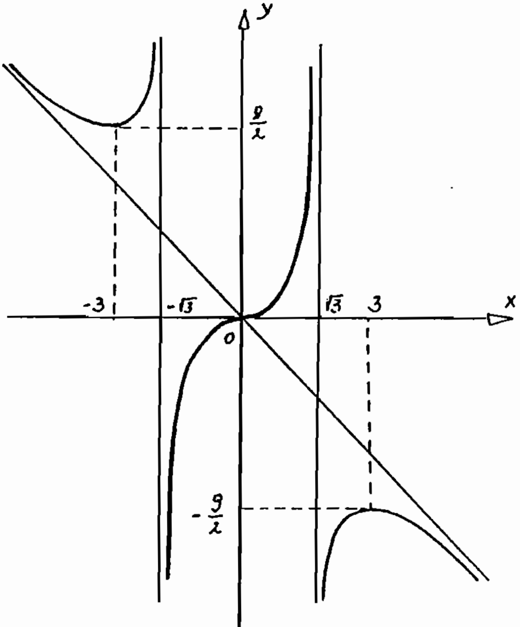
$y'(x) = \frac{x^2(9-x^2)}{(3-x^2)^2}$ kako je $(3-x^2)^2 \geq 0$ – ovaj izraz ne uzimamo u obzir jer ne može promjeniti znak od y'.

	$-\infty$ … -3	-3 … 0	0 … $+3$	$+3$ … $+\infty$
$3+x$	− 0	+	+	+
x	−	− 0	+	+
$3-x$	+	+	+ 0	−
	+	−	+	−
	↗	↘	↗	↘

10° Skica:



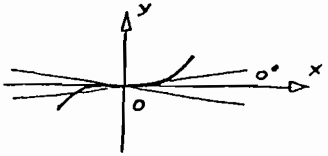
11° Grafik f-je:



* Mogli smo ispitati još: uglove ulaska i izlaska u x=0, u tačkama ekstrema, provjeriti presjeke asimptota s krivom i sl.. Provjerimo samo uglove ulaska u x=0.

$$\lim_{x\to 0_-}\frac{f(x)-f(a)}{x-a} = \lim_{x\to 0_-}\frac{\frac{x^3}{3-x^2}-0}{x-0} = \lim_{x\to 0_-}\frac{x^3}{x(3-x^2)} = \lim_{x\to 0_-}\frac{x^3}{x(3-x^2)} =$$

$$\lim_{x\to 0_-}\frac{x^2}{(3-x^2)} = \lim_{x\to 0_-}\frac{1}{\frac{3}{x^2}-1} = \frac{1}{+\infty} = \underline{0}. \quad tg\,\alpha = 0 \Rightarrow \underline{\alpha^\circ = 0^\circ}.$$



$$\lim_{x\to 0_+}\frac{1}{\frac{3}{x^2}-1} = \frac{1}{+\infty} = 0_+ \Rightarrow \underline{\alpha^\circ = 0^\circ}$$

(Z) Nacrtati grafik f-je: $y=\dfrac{x^4}{x^3-1}$.

RJ:

1° $D(f)$: $x^3-1\neq 0 \Rightarrow x^3\neq 1 \Rightarrow x\neq 1$. F-ja nije definisana za $x=1$. Vjerovatno će to biti horizontalna asimptota!

2° Simetričnost:

$y(-x)=\dfrac{x^4}{-x^3-1}\neq -y(x), \neq y(x)$ pa f-ja nije ni parna ni neparna i nije simetrična

3° Nule f-je:

$y=0$ za $x=0$ - četverostruka nula. Ova f-ja prolazi kroz koordinatni početak.

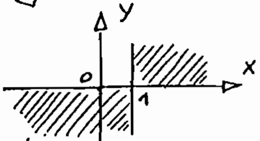
4° Znak f-je:

$y=\dfrac{x^4}{x^3-1}$. Faktor koji mijenja znak je $x^3-1=0$

TABLICA:

	$-\infty$	1	$+\infty$
x^3-1	$-$	$+$	
	$-$	$+$	

SKICA:



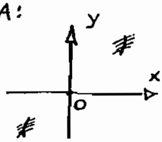
5° Da malo ispitamo definiciono područje:

Tačke: $(\pm\infty)$.

$$\lim_{x\to+\infty} f(x)=\lim_{x\to+\infty}\frac{x^4}{x^3-1}\cong\lim_{x\to+\infty}\frac{x^4}{x^3}=\lim_{x\to+\infty} x=+\infty$$

$$\lim_{x\to-\infty} f(x)=\lim_{x\to-\infty} x=-\infty$$

SKICA:

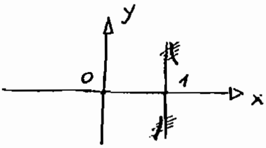


6° Asimptote:

F-ja nije definisana u tački $x=1$, pa onda je moguće da je to V.AS.

$$\lim_{x\to 1_-} f(x)=\lim_{x\to 1_-}\frac{x^4}{x^3-1}=\frac{1}{0_-}=-\infty$$

$$\lim_{x\to 1_+} f(x)=\lim_{x\to 1_+}\frac{x^4}{x^3-1}=\frac{1}{0_+}=+\infty$$



kosa asimptota (k.AS):

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{x^4}{x(x^3-1)}=\lim_{x\to+\infty}\frac{x^4}{x^4(1-\frac{1}{x^3})}=\frac{1}{1-0_+}=1.$$

$$\lim_{x\to-\infty}\frac{f(x)}{x}=\lim_{x\to-\infty}\frac{x^4}{x^4(1-\frac{1}{x^3})}=\lim_{x\to-\infty}\frac{1}{1-\frac{1}{x^3}}=\frac{1}{1-0_-}=1.$$

$\Rightarrow a=1$

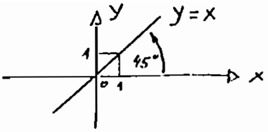
$y=ax+b$ k.AS.

$$\lim_{x\to+\infty}(f(x)-x)=\lim_{x\to+\infty}\left(\frac{x^4}{x^3-1}-x\right)=\lim_{x\to+\infty}\frac{x^4-x^4+1}{x^3-1}=\lim_{x\to+\infty}\frac{1}{x^3-1}=\frac{1}{+\infty}=0_+.$$

$$\lim_{x\to-\infty}(f(x)-x)=\lim_{x\to-\infty}\frac{1}{x^3-1}=\frac{1}{-\infty}=0.$$

$\Rightarrow b=0 \Rightarrow y=x$ KAS.

Imamo sljedeću skicu:



7° Izvodi:

$$y_{(x)}=\frac{x^4}{x^3-1}$$

$$y'_{(x)}=\left(\frac{x^4}{x^3-1}\right)'=\frac{(x^4)'(x^3-1)-(x^3-1)'\cdot x^4}{(x^3-1)^2}=\frac{4x^3(x^3-1)-x^4(3x^2)}{(x^3-1)^2}=\frac{4x^6-4x^3-3x^6}{(x^3-1)^2}=$$

$$=\frac{x^6-4x^3}{(x^3-1)^2}=\frac{x^3(x^3-4)}{(x^3-1)^2}$$

$$y''_{(x)}=\frac{(x^6-4x^3)'(x^3-1)-((x^3-1)^2)'(x^6-4x^3)}{(x^3-1)^4}=\frac{(6x^5-12x^2)(x^3-1)^2-(2(x^3-1)3x^2\cdot(x^6-4x^3))}{(x^3-1)^4}=$$

$$=\frac{(6x^5-12x^2)(x^3-1)^2-(2(x^3-1)3x^2\cdot(x^6-4x^3))}{(x^3-1)^4}=\frac{\cancel{(x^3-1)}((6x^5-12x^2)(x^3-1)-6x^2(x^6-4x^3))}{(x^3-1)^{\cancel{4}\,3}}=$$

$$=\frac{6x^8-18x^5+12x^2-6x^8+24x^5}{(x^3-1)^3}=\frac{6x^5+12x^2}{(x^3-1)^3}=\frac{6x^2(x^3+2)}{(x^3-1)^3}$$

$$y'_{(x)}=\frac{x^3(x^3-4)}{(x^3-1)^2}$$

1) $D(f')$: $x^3\neq 1 \Rightarrow x\neq 1$ – nije EKSTREM jer f-ja nije definisana u $x=1$.

2) Nule: $x=0$ – trostruka
$x^3-4=0 \Rightarrow x^3=4 \Rightarrow x=\sqrt[3]{4}$ }
ST. I vrste, mogući ekstremi.

$$y''_{(x)}=\frac{6x^2(x^3+2)}{(x^3-1)^3}$$

1) $D(f'')$: $x^3-1\neq 0 \Rightarrow x\neq 1$ – nije prevoj.

2) Nule: $x=0$ – dvostruka
$x^3+2=0 \Rightarrow x=-\sqrt[3]{2}$
Ovo je mogući prevoj.

8° Extremi:

$$y''_{(x)}=\frac{6x^2(x^3+2)}{(x^3-1)^3}\quad,\quad ST.\ I:\ x=\sqrt[3]{4}\cong 1{,}59$$

$$y''(\sqrt[3]{4})=\frac{6\cdot(\sqrt[3]{4})^2(4+2)}{(4-1)^3}=\frac{>0}{>0}>0 \Rightarrow \text{minimum}$$

$$y(\sqrt[3]{4})=\frac{6{,}5}{3}=\underline{2{,}12} \Rightarrow \underline{Ext.\ 1.\ (1{,}59,\ 2{,}12)}$$

$y''(0)=0$ – zahtjeva provjeru višim izvodima.

9° Ispitajmo monotonost f-je:

$$y'_{(x)}=\frac{x^3(x^3-4)}{\underbrace{(x^3-1)^2}_{>0}} \Rightarrow \text{nule su: } x=0,\ x=\sqrt[3]{4}=1{,}59$$

	$-\infty$ … 0	0 … 1,59	1,59 … $+\infty$
x	−	+	+
(x^3-4)	−	−	+
	(+)	(−)	(+)
	↗	↘	↗

$$y''(x) = \frac{6x^2(x^3+2)}{(x^3-1)^3} \Rightarrow \text{nule}: x=0 \to \text{ali ne utiče na znak}$$

$x = -\sqrt[3]{2} \cong -1{,}26$, $x=1$

	$-\infty$ … $-1{,}26$	$-1{,}26$ … 1	1 … $+\infty$
x^3+2	−	+	+
x^3-1	−	−	+
	+	−	+
	„U"	„∩"	„U"
	$y''>0$	$y''<0$	$y''>0$

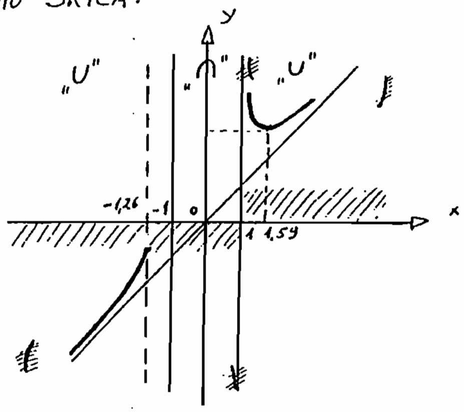
$x=1$ - nije prevoj jer f-ja nije definisana u $x=1$.

$x=-1{,}26$ - je prevoj.

$y(-1{,}26) = \frac{2{,}519}{-3} = -0{,}8399.$

Prev. 1. $(-1{,}26, -0{,}8399)$

10° SKICA:



Tačka nula je vjerovatno ekstrem jer iz prevojne tačke f-ja ide „∩".

Trebali bi provjeriti tačku $x=0$ i to naći više izvode. Međutim, imamo malo komplikovan izraz. Bolje je da malo prostudiramo ono što smo našli do sada.

1) f-ja je u okolini tačke $x=0$ konkavna ∩.

2) raste do $x=0$, a od $x=0$ opada

3) prolazi kroz $x=0$

4) cijelo vrijeme se zadržava ispod x ose.

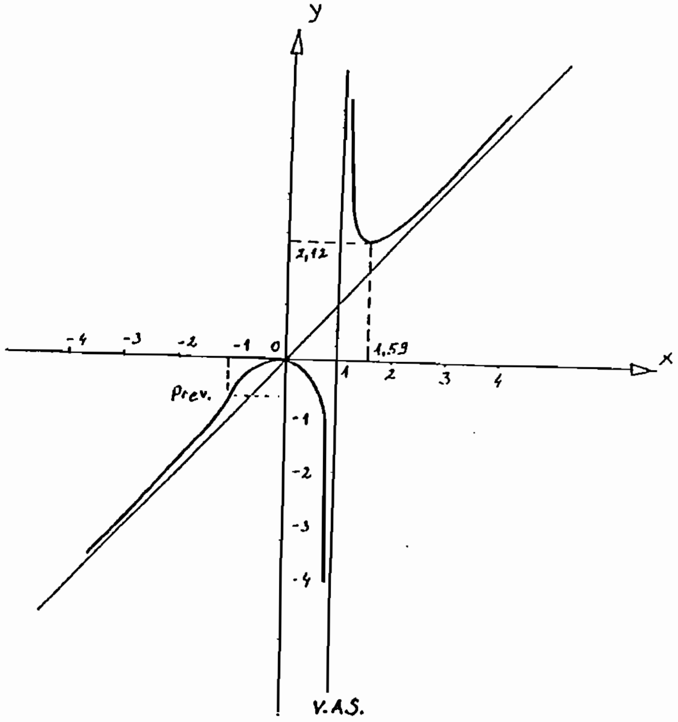
Ovo nam govori da je $x=0$ tačka ekstrema i to maximuma. Ext. 2. $(0,0)$

Možemo uraditi i malu provjeru. U $x=0$ asimptota bi trebala sjeći f-ju u $x=0$, možda i u još nekoj tački ali u $x=0$ sigurno.

$$\frac{x^4}{x^3-1} = x \,/\cdot(x^3-1)\,,\ x\neq 1 \Rightarrow x^4 = x(x^3-1) \Rightarrow x^4 - x^4 + x = 0 \Rightarrow \boxed{x=0}\ (T)$$

Samo u $x=0$ asimptota sječe f-ju.

11° Grafik f-je:



(Z) Ispitati tok f-je, i nacrtati krivu Liniju:

$$y = \frac{x-2}{\sqrt{x^2+1}}$$

RJ:

1° $D(f(x))$: a) $\sqrt{x^2+1} \neq 0$
b) $x^2+1 \geq 0$ $\Rightarrow \forall x \in \mathbb{R}$

2° Simetrija:

$y(-x) = \frac{-x-2}{\sqrt{x^2+1}} \neq -y(x), \neq y(x)$ pa f-ja nije simetrična

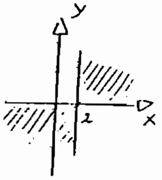
3° Nule: $y=0$ za $x-2=0 \Rightarrow \underline{x=2}$

$y=0$ za $x=2$

$y(0) = -2$ – prolaz kroz y osu.

4° Znak: $x-2=0 \Rightarrow x=2 \Rightarrow$

	$-\infty$	2	$+\infty$
$x-2$	$-$	0	$+$
	$\ominus$		$\oplus$



5° Izvodi:

$$y_{(x)} = \frac{x-2}{\sqrt{x^2+1}}$$

$$y'_{(x)} = \left(\frac{x-2}{\sqrt{x^2+1}}\right)' = \frac{(x-2)'\sqrt{x^2+1} - \left(\sqrt{x^2+1}\right)'(x-2)}{(x^2+1)} = \frac{\sqrt{x^2+1} - \left(\frac{1}{2}(x^2+1)^{\frac{1}{2}-1}\cdot 2x(x-2)\right)}{(x^2+1)} =$$

$$= \frac{\sqrt{x^2+1} - \left(\frac{1}{2}\cdot\frac{2x(x-2)}{\sqrt{x^2+1}}\right)}{(x^2+1)} = \frac{\frac{x^2+1-x^2+2x}{\sqrt{x^2+1}}}{x^2+1} = \underline{\frac{1+2x}{(x^2+1)\sqrt{x^2+1}}}.$$

$$y''_{(x)} = \left(\frac{1+2x}{(x^2+1)\sqrt{x^2+1}}\right)' = \left(\frac{1+2x}{(x^2+1)^{3/2}}\right)' = \frac{(1+2x)'(x^2+1)^{3/2} - \left((x^2+1)^{3/2}\right)'(1+2x)}{(x^2+1)^3} =$$

$$= \frac{2(x^2+1)^{3/2} - (1+2x)\left(\frac{3}{2}(x^2+1)^{3/2-\frac{2}{2}}\cdot 2x\right)}{(x^2+1)^3} = \frac{2(x^2+1)^{3/2} - (1+2x)\cdot 3x(x^2+1)^{1/2}}{(x^2+1)^3} =$$

$$= \frac{2(x^2+1)^{\frac{1}{2}}(x^2+1) - (1+2x)\cdot 3x(x^2+1)^{\frac{1}{2}}}{(x^2+1)^3} = \frac{(x+1)^{\frac{1}{2}}\left(2(x^2+1) - 3x(1+2x)\right)}{(x^2+1)^3} =$$

$$= \frac{\sqrt{(x^2+1)}\cdot(2x^2+2-3x-6x^2)}{(x^2+1)^3} = \underline{\frac{\sqrt{(x^2+1)}\cdot(-4x^2-3x+2)}{(x^2+1)^3}}.$$

$$y'_{(x)} = \frac{1+2x}{(x^2+1)\sqrt{x^2+1}}$$

1) $D(f'(x))$: $x^2+1 \neq 0 \quad \forall x\in\mathbb{R}$; $x^2+1 \geq 0 \quad \forall x\in\mathbb{R}$; $\sqrt{x^2+1} \neq 0 \quad \forall x\in\mathbb{R}$ $\Big\}$ $\forall x \in \mathbb{R}$.

2) Nule: $y'_{(x)} = 0$ za $1+2x = 0 \Rightarrow$
$x = -\frac{1}{2}$ ST. I - mogući ekstrem

$$y''_{(x)} = \frac{\sqrt{(x^2+1)}(-4x^2-3x+2)}{(x^2+1)^3}$$

1) $D(f'')$: $(x^2+1)^3 \neq 0 \Rightarrow \forall x\in\mathbb{R}$

2) Nule: $-4x^2-3x+2=0$

$$x_{1,2} = \frac{+3\pm\sqrt{9+32}}{-8} = \frac{+3\mp\sqrt{41}}{-8}$$

$$x_1 = \frac{3-\sqrt{41}}{-8} \cong \underline{0,43}.$$

$$x_2 = \frac{3+\sqrt{41}}{-8} \approx \underline{-1,175}.$$

6° Ispitajmo definiciono područje:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{x-2}{\sqrt{x^2+1}} = \lim_{x\to+\infty}\frac{x-2}{\sqrt{x^2(1+\frac{1}{x^2})}} = \lim_{x\to+\infty}\frac{x-2}{|x|\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to+\infty}\frac{x-2}{|x|\cdot 1} =$$

$$= \lim_{x\to+\infty}\frac{x-2}{|x|} = \frac{x}{|x|} = \operatorname{sgn}x\cdot\frac{x}{x} = \underline{1}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{x-2}{|x|\cdot 1} = \frac{x}{|x|} = \underline{-1}.$$

$\Big\}$ Imamo dvije H.AS. $y=1$, $\underline{y=-1}$.

7° Asimptote (KAS. i V.AS).

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{x-2}{x\sqrt{x^2+1}} = \lim_{x\to+\infty}\frac{x}{x\sqrt{x^2+1}} - \lim_{x\to+\infty}\frac{2}{x\sqrt{x^2+1}} =$$

$$= \lim_{x\to+\infty} \frac{1}{\sqrt{x^2+1}} - \lim_{x\to+\infty} \frac{2}{x\sqrt{x^2+1}} = \infty - \infty = \left| z.R.T. \right| = \lim_{x\to+\infty} \frac{x-2}{x\sqrt{x^2+1}} =$$

$$= \lim_{x\to+\infty} \frac{x-2}{x\cdot|x|\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to+\infty} \frac{x-2}{x\cdot|x|} \approx \frac{x}{x|x|} \cong \frac{1}{|x|} = \underline{0}.$$

$\lim_{x\to-\infty} \frac{1}{|x|} = \underline{0}.$ Nema kose asimptote i pokazuje na horizontalnu (mislim upućuje).

$y'_{(x)} = \frac{1+2x}{(x^2+1)\sqrt{x^2+1}}$, kandidat za ekstrem je $x = -\frac{1}{2}$.

$$y''_{(x)} = \frac{\sqrt{x^2+1}\cdot(-4x^2-3x+2)}{(x^2+1)^3} \quad ; \quad y''(-\tfrac{1}{2}) = \frac{(>0)\cdot(-4\cdot(\frac{1}{4})+\frac{3}{2}+2)}{\underbrace{(\frac{1}{4}+1)^3}_{>0}} = -1+2+\tfrac{3}{2} > 0$$

$\Rightarrow$ da imamo minimum.

$$y(-\tfrac{1}{2}) = \frac{-\frac{1}{2}-2}{\sqrt{\frac{1}{4}+1}} = \frac{-\frac{5}{2}}{\sqrt{\frac{5}{4}}} = \frac{-5}{\sqrt{5}} \cong \underline{-2{,}236} \Rightarrow \underline{Ext.1.\,(-0{,}5,\ -2{,}236)}$$

9° Prevojne tačke

kandidati su: $x_1 = 0{,}43$, $x_2 = -1{,}175$.

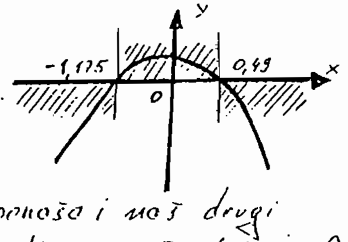
Njih možemo ispitati test tačkama koje uzimamo između x_1 i x_2 preko drugog izvoda, ali u ovom slučaju lakše je pogledati tablice monotonosti i konveksnosti.

$$y''_{(x)} = \frac{\sqrt{x^2+1}\,(-4x^2-3x+2)}{\underbrace{(x^2+1)^3}_{>0}}$$

Znak zavisi samo od $-4x^2-3x+2=0$ pa kada je ova f-ja negativna i $y''_{(x)}$ - je negativan.

* Nacrtajmo f-ju $y = -4x^2-3x+2$

Prevojne tačke su nule ove f-je. Treba samo otkriti da li je otvorena prema gore ili prema dole. To ćemo urediti drugim izvodom.



$y'_{(x)} = -8x-3$

$y''_{(x)} = -8 < 0 \Rightarrow$ „∩" konkavna.

Vidimo da f-ja dva puta mijenja predznak. Tako se ponaša i naš drugi izvod. kad je ova f-ja negativna, naša f-ja je „∩" -konkavna. kad je ova f-ja pozitivna naša f-ja je „U" konveksna, a sve ovo potvrđuje da su x_1 i x_2 prevojne tačke. Ostaje da izračunamo koordinate

prevojnih tačaka:

$y(0{,}43) = \frac{x-2}{\sqrt{x^2+1}} = \frac{0{,}42-2}{\sqrt{0{,}42^2+1}} = -1{,}46$. Prev. 1. $(0{,}43, -1{,}46)$

$y(-1{,}175) = \frac{-3{,}175}{\sqrt{(-1{,}175)^2+1}} = -2{,}06$. Prev. 2. $(-1{,}175, -2{,}06)$

10° Monotonija:

$y'(x) = \frac{1+2x}{(x^2+1)\sqrt{x^2+1}} \Rightarrow$ $y'(x) > 0$ za $x > -\frac{1}{2}$ „↗"

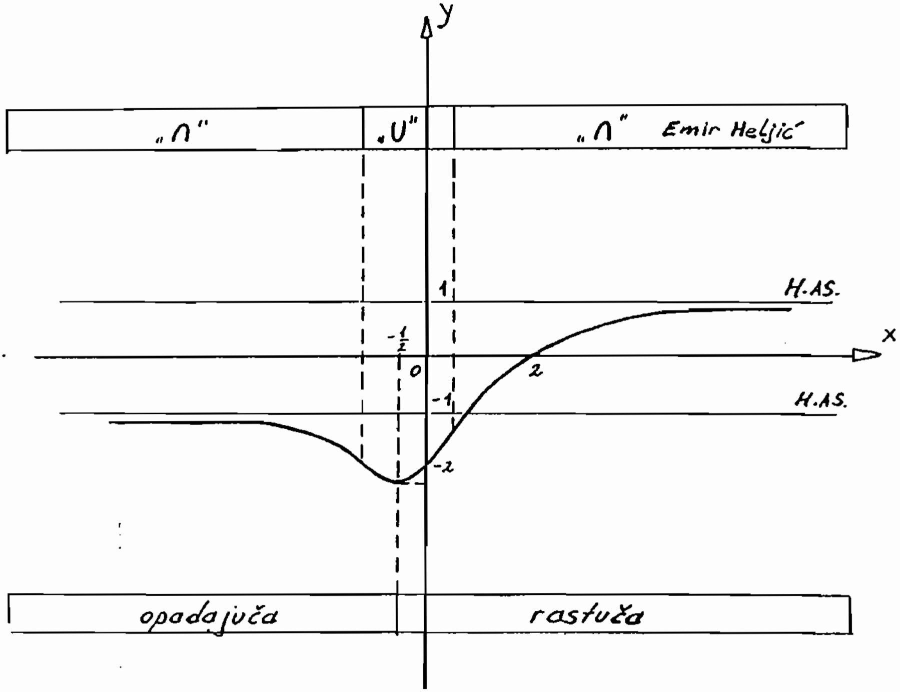
$y'(x) < 0$ za $x < -\frac{1}{2}$ „↙"

11° Nule f-je:

$x = 2$

$y(0) = -\frac{2}{1} = -2$

12° Grafik f-je:



(Z) Nacrtati grafik f-je : $f(x) = 1 - x + \sqrt{\frac{x^3}{x+3}}$.

RJ:

1° Definiciono područje:

a) $\sqrt{\frac{x^3}{x+3}} \Rightarrow \frac{x^3}{x+3} \geq 0$
b) $x+3 \neq 0 \Rightarrow x \neq -3$
} $\Rightarrow$ TABLICA:

	$-\infty$ … -3	-3 … 0	0 … $+\infty$
$x+3$	$-$	$+$	$+$
x^3	$-$	$-$	$+$
	$+$	$-$	$+$

$D(f): x \in (-\infty, -3) \cup [0, +\infty)$

2° Simetričnost:

$y(-x) = 1 + x + \sqrt{\frac{-x^3}{-x+3}} \neq -y(x), \neq y(x)$ F-ja nije simetrična!

3° Nule f-je:

$y = 0$ za $1 - x + \sqrt{\frac{x^3}{x+3}} = 0 \Rightarrow \sqrt{\frac{x^3}{x+3}} = (x-1) /^2 \Rightarrow \frac{x^3}{x+3} = (x^2 - 2x + 1)$

$\frac{x^3}{x+3} = (x^2 - 2x + 1) / \cdot (x+3), \; x \neq -3 \Rightarrow x^3 + 3x^2 - 2x^2 - 6x + x + 3 = x^3 \Rightarrow$

$3x^2 - 2x^2 - 6x + x + 3 = 0 \Rightarrow \underline{x^2 - 5x + 3 = 0} \Rightarrow x_{1,2} = \frac{5 \pm \sqrt{25 - 4 \cdot 3}}{2} \Rightarrow$

$x_{1,2} = \frac{5 \pm \sqrt{13}}{2}$.

$x_1 = \frac{5 - \sqrt{13}}{2} = \underline{0{,}697}$.

$x_2 = \frac{5 + \sqrt{13}}{2} = \underline{4{,}302}$.

Potrebno je provjeriti nule:

$x_1 = 0{,}697 \Rightarrow 1 - 0{,}697 + \sqrt{\frac{(0{,}697)^3}{3{,}697}} = 0{,}303 + 0{,}3026 \cong 0{,}6$

x_1 nije nula f-je!

$x_2 = 4{,}3 \Rightarrow 1 - 4{,}3 + \sqrt{\frac{(4{,}3)^3}{7{,}3}} = -3{,}3 + 3{,}3 = \underline{0}$. - ovo je nula f-je.

4° Izvodi:

$y'(x) = \left(1 - x + \sqrt{\frac{x^3}{x+3}}\right)' = 0 - 1 + \left(\sqrt{\frac{x^3}{x+3}}\right)' = -1 + \frac{2x^3 + 9x^2}{2(x+3)^2 \sqrt{\frac{x^3}{x+3}}}$

$* \left(\sqrt{\frac{x^3}{x+3}}\right)' = \frac{1}{2}\left(\frac{x^3}{x+3}\right)^{-\frac{1}{2}} \cdot \left(\frac{3x^2 \cdot (x+3) - x^3 \cdot 1}{(x+3)^2}\right) = \frac{1}{2\sqrt{\frac{x^3}{x+3}}} \cdot \frac{2x^3 + 9x^2}{(x+3)^2} =$

$= \frac{2x^3 + 9x^2}{2(x+3)^2 \sqrt{\frac{x^3}{x+3}}}$

$y''(x) = \left(\frac{2x^3 + 9x^2}{2(x+3)^2 \sqrt{\frac{x^3}{x+3}}} - 1\right)' = \frac{1}{2}\left(\frac{x^2(2x+9)}{(x+3)^2 \sqrt{\frac{x^3}{x+3}}}\right)' = \frac{1}{2}\left(\frac{x^2(2x+9)}{\sqrt{x^3(x+3)^3}}\right)' =$

$= \frac{1}{2} \frac{(6x^2 + 18x)\sqrt{(x(x+3))^3} - x^2(2x+9)\left(\frac{3}{2}(x(x+3))^{1/2} \cdot (2x+3)\right)}{x^3(x+3)^3} =$

$= \frac{1}{2} \frac{6x(x+3)(x+3) \cdot x\sqrt{x(x+3)} - \frac{3}{2}x^2(2x+9)\sqrt{x(x+3)}(2x+3)}{x^3(x+3)^3} =$

$$= \frac{1}{2}\,\frac{x^2\sqrt{x(x+3)}\,\bigl(12(x^2+6x+9)-3(4x^2+6x+18x+27)\bigr)}{2x^3(x+3)^3} =$$

$$= \frac{1}{4}\,\frac{x^2\sqrt{x(x+3)}\cdot\bigl(12x^2+72x+108-12x^2-72x-81\bigr)}{x^3(x+3)^3} = \underline{\frac{27x^2\sqrt{x(x+3)}}{4x^3(x+3)^3}}.$$

$$y''_{(x)} = \left(\frac{2x^3+9x^2}{2(x+3)^2\sqrt{\frac{x^3}{x+3}}}-1\right)' = \frac{1}{2}\left(\frac{2x^3+9x^2}{(x+3)^2\sqrt{\frac{x^3}{x+3}}}\right)' =$$

$$= \frac{1}{2}\,\frac{6x(x+3)(x+3)^2\sqrt{\frac{x^3}{x+3}}-(2x^3+9x^2)\left((x+3)^2\sqrt{\frac{x^3}{x+3}}\right)'}{(x+3)^4\left(\sqrt{\frac{x^3}{x+3}}\right)^2} =$$

$$= \frac{6x(x+3)^3\sqrt{\frac{x^3}{x+3}}-x^2(2x+9)\left(2(x+3)\sqrt{\frac{x^3}{x+3}}+(x+3)^2\cdot\frac{1}{2}\left(\frac{x^3}{x+3}\right)^{-\frac{1}{2}}\cdot\left(\frac{3x^2(x+3)-x^3}{(x+3)^2}\right)\right)}{2\cdot(x+3)^4\,\frac{x^3}{(x+3)}} =$$

$$= \frac{6x(x+3)^3\sqrt{\frac{x^3}{x+3}}-x^2(2x+9)\left(2(x+3)\sqrt{\frac{x^3}{x+3}}+\frac{1}{2}\,\frac{(x+3)^2(3x^3-x^3+9x^2)}{\sqrt{\frac{x^3}{x+3}}\,(x+3)^2}\right)}{2\cdot(x+3)^3x^3} =$$

$$= \frac{1}{2}\,\frac{6x(x+3)^3\sqrt{\frac{x^3}{x+3}}-x^2(2x+9)\left(2(x+3)\sqrt{\frac{x^3}{x+3}}+\frac{1}{2}\,\frac{2x^3+9x^2}{\sqrt{\frac{x^3}{x+3}}}\right)}{x^3(x+3)^3} =$$

$$= \frac{1}{2}\cdot\frac{6x(x+3)^3\sqrt{\frac{x^3}{x+3}}-x^2(2x+9)\left(\frac{4(x+3)\frac{x^3}{(x+3)}+2x^3+9x^2}{2\cdot\sqrt{x^3/(x+3)}}\right)}{x^3(x+3)} =$$

$$= \frac{1}{2}\,\frac{6x(x+3)^3\sqrt{\frac{x^3}{x+3}}-x^2(2x+9)\cdot\left(\frac{4x^3+2x^3+9x^2}{2\sqrt{\frac{x^3}{x+3}}}\right)}{x^3(x+3)^3} =$$

$$= \frac{1}{2}\,\frac{\frac{12x(x+3)^2x^3-x^2(2x+9)x^2(6x+9)}{2\cdot\sqrt{\frac{x^3}{x+3}}}}{x^3(x+3)^3} = \frac{12x^4(x+3)^2-3x^4(2x+9)(2x+3)}{4\sqrt{\frac{x^3}{x+3}}\cdot x^3(x+3)^3} =$$

$$= \frac{3x^4\bigl(4(x^2+6x+9)-(4x^2+6x+18x+27)\bigr)}{4\sqrt{\frac{x^3}{x+3}}\cdot x^3(x+3)^3} = \frac{27x^4}{4x^3(x+3)^3\sqrt{\frac{x^3}{x+3}}} = \underline{\underline{\frac{27x}{4(x+3)^3\sqrt{\frac{x^3}{x+3}}}}}.$$

$$y'_{(x)} = \frac{2x^3+9x^2}{2(x+3)^2\sqrt{\frac{x^3}{x+3}}}-1$$

$D(f')$: 1) $x+3\neq 0 \Rightarrow x\neq -3$

2) $\frac{x^3}{x+3}\geq 0 \Rightarrow x\in(-\infty,-3)\cup[0,+\infty)$

3) $\sqrt{\frac{x^3}{x+3}}\neq 0 \Rightarrow x\neq 0$

$\Bigg\}\; x\in(-\infty,-3)\cup(0,+\infty).$

1.

$y'(x)=0$ za $\frac{2x^3+9x^2}{2(x+3)^2\sqrt{\frac{x^3}{x+3}}}-1=0 \Rightarrow 2x^3+9x^2=2(x+3)^2\sqrt{\frac{x^3}{x+3}} \Rightarrow$

$(2x+9)x^2=2(x+3)^2\sqrt{\frac{x^3}{x+3}}\ /^2 \Rightarrow x^4(2x+9)^2=4(x+3)^4\frac{x^3}{(x+3)} \Rightarrow$

$x(2x+9)^2=4(x+3)^3 \Rightarrow x(4x^2+36x+81)=4(x^2+6x+9)(x+3) \Rightarrow$

$4x^3+36x^2+81x=(4x^2+24x+36)(x+3) \Rightarrow 4x^3+36x^2+81x=4x^3+12x^2+24x^2+72x+36x+$
$+108 \Rightarrow 27x+108=0 \Rightarrow \underline{x=-4}$. ST. I – stacionarna tačka.

$$y''(x)=\frac{27}{4(x+3)^2\sqrt{x(x+3)}}$$

1° $D(f''(x))$: 1) $x+3\neq 0 \Rightarrow x\neq -3$

2) $\sqrt{x(x+3)}\neq 0 \Rightarrow x\neq 0,\ x\neq -3$

3) $x(x+3)\geq 0 \Rightarrow x\in(-\infty,-3]\cup[0,+\infty)$

$D(f''(x))$: $x\in(-\infty,-3)\cup(0,+\infty)$.

2° $y''(x)=0$ – nikada – ni za jedno $x\in\mathbb{R}$.

Nemamo mogućih prevoja u definicionom području.

5° Ispitajmo karakteristične tačke def. područja:

$x=\pm\infty,\ x=0,\ x=-3$.

$T(+\infty)$:

$$\lim_{x\to+\infty}\left(1-x+\sqrt{\frac{x^3}{x+3}}\right)=\lim_{x\to+\infty}\left(\frac{\sqrt{x+3}(1-x)+\sqrt{x^3}}{\sqrt{x+3}}\right)=\lim_{x\to+\infty}\frac{(x+3)(1-x)^2-x^3}{\sqrt{x+3}\left(\sqrt{x+3}(1-x)-\sqrt{x^3}\right)}=$$

$$=\lim_{x\to+\infty}\frac{(x+3)(1-2x+x^2)-x^3}{\sqrt{x+3}\left(\sqrt{x+3}(1-x)-\sqrt{x^3}\right)}=\lim_{x\to+\infty}\frac{x-2x^2+x^3+3-6x+3x^2-x^3}{\sqrt{x+3}\left(\sqrt{x+3}(1-x)-\sqrt{x^3}\right)}=$$

$$=\lim_{x\to+\infty}\frac{x^2-5x+3}{(x+3)(1-x)-\sqrt{x^3(x+3)}}=\lim_{x\to+\infty}\frac{x^2-5x+3}{x-x^2+3-3x-\sqrt{x^4+3x^3}}=$$

$$=\lim_{x\to+\infty}\frac{x^2-5x+3}{-x^2-2x+3-\sqrt{x^4+3x^3}}=\lim_{x\to+\infty}\frac{x^2-5x+3}{-x^2-2x+3-\sqrt{x^4+3x^3}}=$$

$$=\lim_{x\to+\infty}\frac{1-\frac{5}{x}+\frac{3}{x^2}}{-1-\frac{2}{x}+\frac{3}{x^2}-\sqrt{1+\frac{3}{x}}}=\frac{1}{-1-1}=\underline{-\frac{1}{2}}.\qquad \text{H.AS.}$$

$T(-\infty)$:

$$\lim_{x\to-\infty}\frac{x^2-5x+3}{-x^2-2x+3-\sqrt{x^4+3x^3}}=\lim_{x\to-\infty}\frac{1-\frac{5}{x}+\frac{3}{x^2}}{-1-\frac{2}{x}+\frac{3}{x^2}-\sqrt{1+\frac{3}{x}}}=\frac{1}{-1-1}$$

E ovo ne može ovako!

$$\lim_{x\to-\infty}\left(1-x+\sqrt{\frac{x^3}{x+3}}\right) = 1+\infty+\lim_{x\to-\infty}\sqrt{\frac{x^3}{x+3}} = \lim_{x\to-\infty}\frac{\frac{3}{2}x^{\frac{1}{2}}}{\frac{1}{2}(x+3)^{-\frac{1}{2}}} =$$

$$= \lim_{x\to-\infty} 2\cdot\frac{3}{2}\sqrt{x}\cdot\sqrt{x+3} = 3\underbrace{\sqrt{x^2+3x}}_{x} = +\infty = 1+\infty+\infty = \underline{+\infty}.$$

$$\sqrt{\lim_{x\to-\infty}\frac{x^3}{x+3}} = \frac{1}{\frac{1}{x^2}+\frac{3}{x^3}} = \underline{+\infty}.$$

$T(0)$:

$$\left.\begin{aligned}\lim_{x\to0_-}\left(1-x+\sqrt{\frac{x^3}{x+3}}\right) &= 1+0+\sqrt{\frac{0_-}{0_-+3}} = \underline{1}.\\ \lim_{x\to0_+}\left(1-x+\sqrt{\frac{x^3}{x+3}}\right) &= 1-0+\sqrt{\frac{0}{0+3}} = \underline{1}.\end{aligned}\right\} \lim_{x\to0}\left(1-x+\sqrt{\frac{x^3}{x+3}}\right) = \underline{1}.$$

$T(-3)$:

$$\lim_{x\to-3_-}\left(1-x+\sqrt{\frac{x^3}{x+3}}\right) = 1+3+\sqrt{\frac{-27}{0_-}} = 1+3+\infty = \underline{+\infty}.$$

$\lim_{x\to-3_+}\left(1+3+\sqrt{\frac{-27}{0_+}}\right)$ – nema limesa jer je ispod korjena $\ominus$.

6° Extremi:

ST. I vrste je: $x=-4$.

$$y''(-4) = \frac{27}{4(-4+3)^2\sqrt{-4(-4+3)}} = \frac{27}{4\cdot1\cdot2} > 0 \Rightarrow \text{maximum}.$$

$$y(-4) = 1+4+\sqrt{\frac{4^3}{1}} = 13 \quad\Rightarrow\quad \boxed{Ext.1.\ (-4, 13)}$$

7° Prevojne tačke i konkavnost i konveksnost:

ST. II vrste – ne postoji ni jedna.

$$y''(x) = \frac{27}{\underbrace{4(x+3)^2}_{>0}\underbrace{\sqrt{x(x+3)}}_{>0}} > 0$$ pa je f-ja: „U" konveksna.

8° Rast i opadanje:

$$y'(x) = \frac{x^2(2x+9)}{2(x+3)^2\sqrt{\frac{x^3}{x+3}}} - 1 = 0 \quad \text{za } \underline{x=-4}.$$

$$y'(-5) = \frac{25\cdot(-10+9)}{2(-2)^2\sqrt{\frac{(-5)^3}{-2}}} - 1 = \frac{-25}{8\cdot7{,}9} < 0 \text{ – opadajuća } \searrow$$

$$y'(-3{,}5) = \frac{(-3{,}5)^2(-7+9)}{2(-0{,}5)^2\sqrt{\frac{(-3{,}5)^3}{-0{,}5}}} - 1 = \frac{24{,}5}{2(>0)(>0)} > 0 \text{ – rastuća } \nearrow$$

$$y'(1) = \frac{1(2+9)}{2\cdot4^2\sqrt{\frac{1}{4}}} - 1 = \frac{11}{16} - 1 < 0 \text{ – opadajuća } \searrow$$

9° Asimptote:

$$\lim_{x\to+\infty}\frac{f(x)}{x}=\lim_{x\to+\infty}\frac{1-x+\sqrt{\frac{x^3}{x+3}}}{x}=\lim_{x\to+\infty}\left(\frac{1}{x}-1+\frac{\sqrt{\frac{x^3}{x+3}}}{x}\right)=$$

$$=0-1+\lim_{x\to+\infty}\frac{\sqrt{\frac{x^3}{x+3}}}{x}=-1+\lim_{x\to+\infty}\sqrt{\frac{x^3}{x^3+3x^2}}=-1+\lim_{x\to+\infty}\sqrt{\frac{x^3}{x^2(x+3)}}=$$

$$=-1+\sqrt{\frac{1}{1+0}}=-1+1=0 \Rightarrow \lim_{x\to+\infty}\frac{f(x)}{x}=0.$$

$$\lim_{x\to-\infty}\frac{f(x)}{x}=-1+\lim_{x\to-\infty}\frac{\sqrt{\frac{x^3}{x+3}}}{x}=-1-\sqrt{\frac{1}{1+\frac{3}{x}}}=-1-1=\underline{-2}.$$

$$\lim_{x\to-\infty}\left(f(x)+2x\right)=\left(1+x+\sqrt{\frac{x^3}{x+3}}\right)=1+\lim_{x\to-\infty}\left(x+\sqrt{\frac{x^3}{x+3}}\right)=$$

$$=1+\lim_{x\to-\infty}\left(x+|x|\sqrt{\frac{x}{x+3}}\right)=1+\lim_{x\to-\infty}\left(x-x\sqrt{\frac{x}{x+3}}\right)=1+\lim_{x\to-\infty}x\left(1-\sqrt{\frac{x}{x+3}}\right)=$$

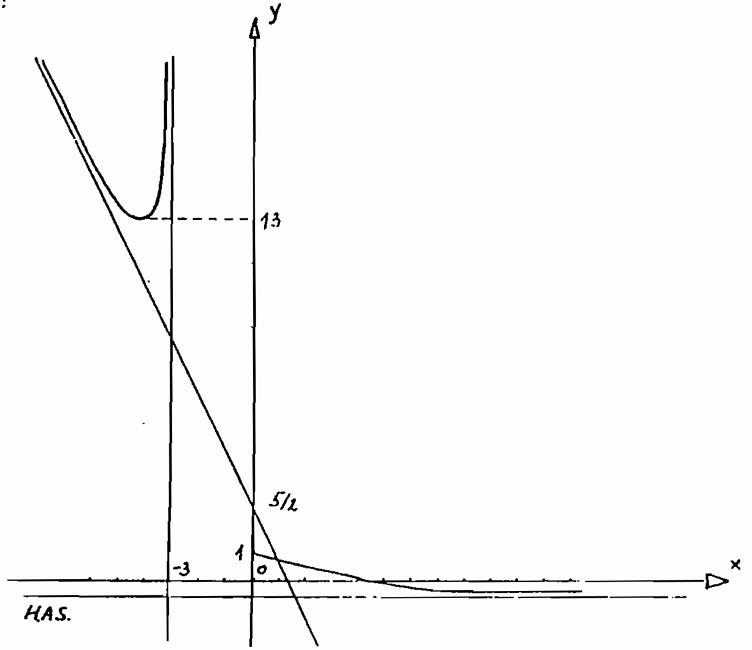
$$=1+\lim_{x\to-\infty}\left(\frac{1-\sqrt{\frac{x}{x+3}}}{\frac{1}{x}}\right)=\left|Lopitol\right|=1+\lim_{x\to-\infty}\frac{-\frac{1}{2}\left(\frac{x}{x+3}\right)^{-\frac{1}{2}}\cdot\left(\frac{x+3-x}{(x+3)^2}\right)}{-\frac{1}{x^2}}=$$

$$=1+\lim_{x\to-\infty}\frac{-\frac{1}{2}\frac{3}{\sqrt{\frac{x}{x+3}}(x+3)^2}}{-\frac{1}{x^2}}=1+\frac{3}{2}\lim_{x\to-\infty}\frac{x^2}{\sqrt{\frac{x}{x+3}}(x+3)^2}=1+\lim_{x\to-\infty}\frac{3\cdot x^2}{2\sqrt{x(x+3)^3}}=$$

$$=1+\frac{3}{2}\cdot 1=\frac{5}{2} \Rightarrow y=ax+b \Rightarrow k.AS. \Rightarrow \boxed{y=-2x+\frac{5}{2}}$$

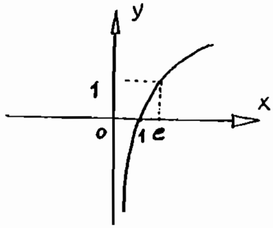
$y=-2x+\frac{5}{2} \Rightarrow y(0)=\frac{5}{2}$, $y=0$ za $x=\frac{5}{4}$.

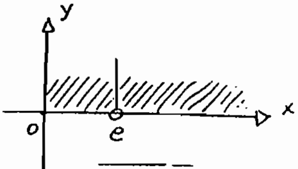
10° Grafik:



(2) Nacrtati grafik f-je: $y = \frac{1}{x(1-\ln x)}$.

* Najprije da nacrtamo f-ju $\ln x$:





1° $D(f(x))$: 1) $x \neq 0$; 2) $1-\ln x \neq 0 \Rightarrow x \neq e$; 3) $\ln x \Rightarrow x > 0$ $\}$ $D(f(x))$: $\forall x > 0 \setminus \{e\}$ $\Rightarrow$ Treba ispitati tačku e.

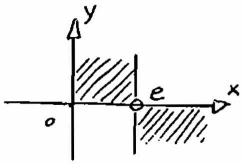
2° Simetrija: $y(-x) = \frac{1}{-x(1-\ln(-x))}$ $\Rightarrow$ nije simetrična f-ja!

3° Nule: nema nula f-je.

4° Znak: $y = \frac{1}{x(1-\ln x)}$

	$-\infty$	e	$+\infty$
$1-\ln x$	+	−	
	+	−	

$1-\ln x > 0$ za $x \in (0,e)$
$1-\ln x < 0$ za $x \in (e,+\infty)$ $\}$ $\Rightarrow$ SKICA:



5° Izvodi:

$$y'_{(x)} = \left[\frac{1}{x(1-\ln x)}\right]' = \frac{1'(x(1-\ln x)) - (x(1-\ln x))' \cdot 1}{x^2(1-\ln x)^2} = \frac{-(x(1-\ln x))'}{x^2(1-\ln x)^2} =$$

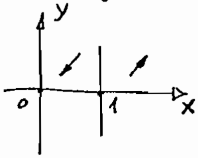
$$= \frac{-(x'(1-\ln x) + (1-\ln x)' \cdot x)}{x^2(1-\ln x)^2} = \frac{-((1-\ln x) + (-\frac{1}{x})x)}{x^2(1-\ln x)^2} = \frac{-((1-\ln x)-1)}{x^2(1-\ln x)^2} =$$

$$= \frac{\ln x}{x^2(1-\ln x)^2}$$

$D(f'(x))$: 1) $x > 0$; 2) $x \neq e$; 3) $x \neq 0$ $\}$ $D(f'(x))$: $\forall x > 0 \setminus \{e\}$

Nule: $\ln x = 0$ za $x = 1$. $x = 1$ je ST. I vrste tj: mogući ekstrem.

$y'(x) = \frac{\ln x}{\underbrace{x^2(1-\ln x)}_{>0}}$ $\Rightarrow$ $\ln x < 0$ za $x \in (0,1)$; $\ln x > 0$ za $x \in (1,+\infty)$ $\}$ $\Rightarrow$



$$y''_{(x)} = \left(\frac{\ln x}{x^2(1-\ln x)}\right)' = \frac{\frac{1}{x}(x^2(1-\ln x)^2) - (2x(1-\ln x)^2 + 2(1-\ln x)(-\frac{1}{x})x^2 \cdot \ln x)}{x^4(1-\ln x)^4} =$$

$$= \frac{\frac{1}{x}(x^2(1-\ln x)^2) - (2x(1-\ln x)^2 + 2(1-\ln x)(-\frac{1}{x})x^2) \cdot \ln x}{x^4(1-\ln x)^4} =$$

$$= \frac{x(1-\ln x)^2 - (2x(1-\ln x)(1-\ln x) - 2x(1-\ln x)) \ln x}{x^4(1-\ln x)} =$$

$$= \frac{x(1-\ln x)^2 - (2x(1-\ln x)(1-\ln x-1)) \cdot \ln x}{x^4(1-\ln x)^4} = \frac{x(1-\ln x)(1-\ln x) - (2x(1-\ln x)(1-\ln x-1)\ln x)}{x^4(1-\ln x)^4} =$$

$$= \frac{x(1-\ln x)(1-\ln x + 2\ln^2 x)}{x^4(1-\ln x)^4} = \frac{2\ln^2 x - \ln x + 1}{x^3(1-\ln x)^3}$$

$$y''_{(x)} = \frac{2\ln^2 x - \ln x + 1}{x^3(1-\ln x)^3}$$

$D(f''_{(x)})$: 1) $x^3 \neq 0 \Rightarrow x \neq 0$
2) $\ln x \Rightarrow x > 0$
3) $1-\ln x \neq 0 \Rightarrow 1 \neq \ln x \Rightarrow x \neq e$.

$\Big\}\ x \in (0,+\infty) \setminus \{e\}$

Stacionarne tačke :

$2\ln^2 x - \ln x + 1 = 0 \Rightarrow \ln x = t \Rightarrow 2t^2 - t + 1 = 0$

$t_{1,2} = \frac{1 \pm \sqrt{1-8}}{4} = \frac{1 \pm \sqrt{-7}}{4}$ nema nule !

Dakle, dobili smo da je , $2\ln^2 x - \ln x + 1 \neq 0$, $\forall x \in \mathbb{R}$.

Ako je : $2\ln^2 x > \ln x - 1$ – tada je $2\ln^2 x - \ln x + 1 > 0$.

Uzmimo nekoliko test tačaka i ispitajmo:

$x = 0{,}5 \Rightarrow 0{,}48 > -0{,}84$ (T)

$x = 2 \Rightarrow 0{,}48 > -0{,}15$ (T)

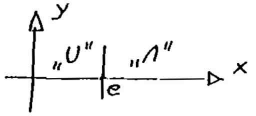
$x = 5 \Rightarrow 2{,}59 > 0{,}30$ (T)

6° Monotonost, konkavnost i konveksnost

$y''_{(x)} < 0$ za $x \in (e,+\infty) \Rightarrow$ „∩"

$y''_{(x)} > 0$ za $x \in (0,e) \Rightarrow$ „∪"

$\Big\} \Rightarrow$

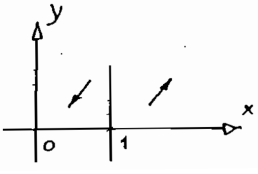


$$y'_{(x)} = \frac{\ln x}{\underbrace{x^2(1-\ln x)^2}_{>0}}$$

$\ln x < 0$ za $x \in (0,1)$

$\ln x > 0$ za $x \in (1,+\infty)$

$\Big\}$ SKICA: $\Rightarrow$



7° Asimptote:

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \frac{1}{x(1-\ln x)} = \frac{1}{-\infty} = \underline{0_-}$. H.AS.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{1}{x(1-\ln x)} = \frac{1}{-\infty} = \underline{0_-}$ - ovaj limes je nepotreban jer f-ja nije definisana za $x<0$.

$\lim\limits_{x\to+\infty} \frac{f(x)}{x} = \lim\limits_{x\to+\infty} \frac{1}{x^2(1-\ln x)} = \lim\limits_{x\to+\infty} \frac{1}{x^2(1-\ln x)} = 0$ – nema k.AS.

$\lim\limits_{x\to 0_+} f(x) = \lim\limits_{x\to 0_+} \frac{1}{x(1-\ln x)} = \underline{+\infty}$.

$\lim\limits_{x\to 0_-} \frac{1}{x} = \underline{-\infty}$ – ne treba ispitivati, jer f-ja nije definisana u $x=0$!

8° Ispitajmo definiciono područje:

$\lim\limits_{x\to e_-} f(x) = \lim\limits_{(e-)} \frac{1}{x(1-\ln x)} = \frac{1}{e\underbrace{(1-1_-)}_{>0}} = \underline{+\infty}$.

$\lim\limits_{x\to e_+} f(x) = \lim\limits_{(e+)} \frac{1}{x(1-\ln x)} = \frac{1}{e\underbrace{(1-1_+)}_{<0}} = \underline{-\infty}$.

} $\underline{\text{V.AS.}}$ $x=e$.

9° Extremi:

ST.I.: $x=1$.

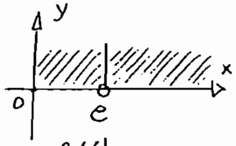
$y''_{(1)} = \frac{2\ln^2 1 - \ln 1 + 1}{1(1)} = 1 > 0 \Rightarrow$ minimum.

$y_{(1)} = \frac{1}{x(1-\ln x)} = \frac{1}{1} = 1$.

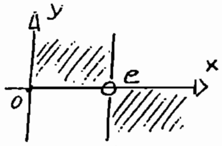
} $\underline{\text{Ext. 1. } (1,1)}$.

10° Prevojnih tačaka nema.

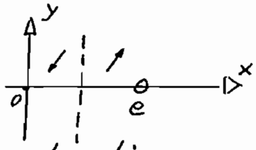
11° Skice:



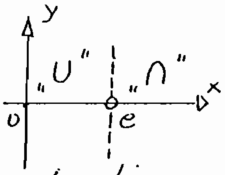
D(f):



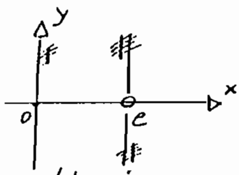
Znak



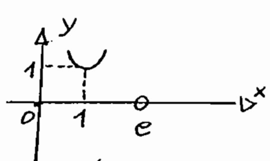
Izvodi



Izvodi

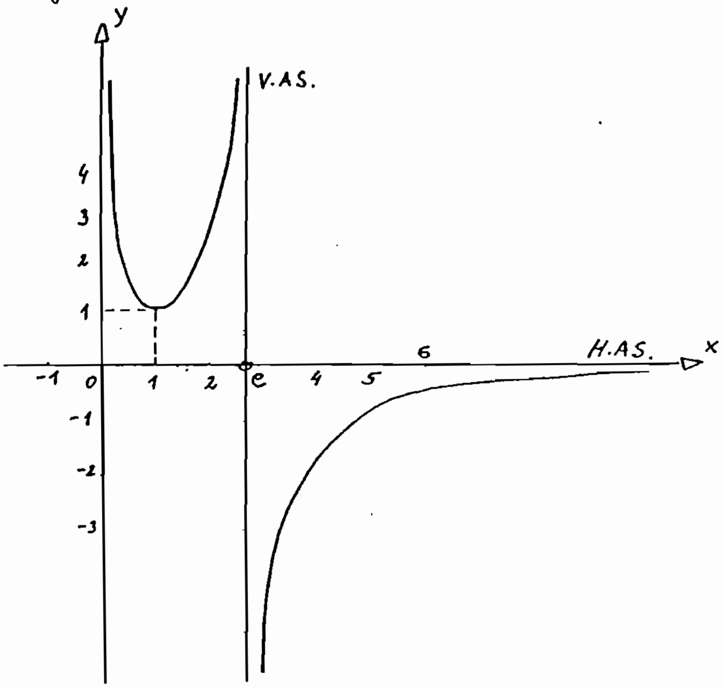


Limesi



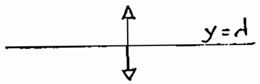
Extrem

12° Grafik:



(Z) Za koje $\lambda \in \mathbb{R}$ jednačina $\lambda x^2 - \ln x - 1 = 0$ ima jedno rješenje, dva i nema rješenja?

* $x > 0$, $\lambda = \frac{1 - \ln x}{x^2}$
* $y = \frac{1 - \ln x}{x^2}$ – treba ispitati grafik ove f-je.
* $y = \lambda \Rightarrow$



Treba uzeti $y = \lambda$ i pomjerati je po y osi i tražiti rješenja.

Ispitajmo sada f-ju: $y = \frac{1 - \ln x}{x^2}$!

1° $D(f)$: $x > 0$

Definiciono područje je nesimetrično pa nema smisla ispitivati simetričnost!

2° F-ja je sastavljena iz $\ln x$, x^2, $(1 - \ln x)$ koje su elementarno neprekidne f-je pa je i kompletna f-ja neprekidna

3° Izvodi:

$$y'_{(x)} = \frac{(1-\ln x)'x^2-(x^2)'(1-\ln x)}{x^4} = \frac{-\frac{1}{x}\cdot x^2-2x(1-\ln x)}{x^4} = \frac{-x-2x+2x\ln x}{x^4} =$$

$$= \frac{-3x+2x\ln x}{x^4} = \frac{x(-3+2\ln x)}{x^4} = \frac{-3+2\ln x}{x^3}, \quad Ex' \equiv Ex.$$

$$y''_{(x)} = \frac{(-3+2\ln x)'x^3-(x^3)'(-3+2\ln x)}{(x^3)^2} = \frac{\frac{2}{x}\cdot x^3-3x^2(-3+2\ln x)}{x^6} =$$

$$= \frac{2x^2+9x^2-6x^2\ln x}{x^6} = \frac{11x^2-6x^2\ln x}{x^6} = \frac{x^2(11-6\ln x)}{x^6} = \frac{11-6\ln x}{x^4}$$

$Ex'' \equiv Ex' \equiv Ex$

4° Ispitajmo definiciono područje:

$T(0_+)$:

$$\lim_{x\to 0_+} \frac{1-\ln x}{x^2} = +\infty \quad \left| \begin{matrix} x=e^t \\ t\to-\infty \end{matrix} \Rightarrow \frac{1-t}{e^{2t}};\ \begin{matrix} t=-x \\ t\to+\infty \end{matrix} \Rightarrow \underset{x\to+\infty}{(1+x)e^{2x}\to+\infty} \right|$$

RJEŠAVANJE LIMESA

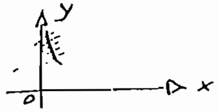
$x=0 \Rightarrow y\to+\infty$

V.AS.

Da vidimo šta još možemo naći vezano za okolinu 0_+:

$y'<0 \Rightarrow$ „↘"

$y''>0 \Rightarrow$ „∪" $\Bigg\} \Rightarrow$

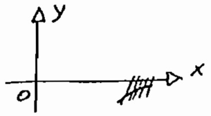


Ispitajmo točku $T(+\infty)$:

$$\lim_{x\to+\infty}\frac{1-\ln x}{x^2} \overset{\frac{\infty}{\infty}}{=} \lim_{x\to+\infty}\frac{1}{x^2} - \lim_{x\to+\infty}\frac{\ln x}{x^2} = \left|\begin{matrix} x=e^t \\ t\to+\infty\end{matrix}\right| = -\lim_{t\to+\infty}\frac{t}{e^{2t}} = \underline{0}.$$

Okolina: $y'>0 \Rightarrow$ „↗"

$O(0_+)$: $y''<0 \Rightarrow$ „∩" $\Bigg\} \Rightarrow$



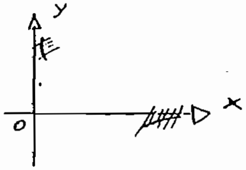
$$\Rightarrow y' = \frac{3-2\ln x}{\underbrace{x^3}_{>0}} = \frac{<0}{>0} < 0$$ – kada uzmemo x-eve bliske 0_+, jer $2\ln x\to-\infty$ kad $x\to 0_+$, i x^3 je pozitivno sve dok se uzimaju x-evi ispred nule tj. do nule, a kada je $x=0$ (, a nije nikad) tada je $x^3=0$.

$$y'' = \frac{\overbrace{11-6\ln x}^{>0}}{\underbrace{x^4}_{>0}} > 0$$

$O(+\infty) \Rightarrow y' = \frac{\overbrace{-3+2\ln x}^{+\infty}}{\underbrace{x^3}_{+\infty}} > 0 \quad ; \quad y'' = \frac{\overbrace{11-6\ln x}^{-\infty, <0}}{\underbrace{x^4}_{+\infty, >0}} < 0$

Pa imamo:

$y' > 0 \Rightarrow \nearrow$, $y' < 0 \Rightarrow \swarrow$; $y'' > 0 \Rightarrow$ „U", $y'' < 0 \Rightarrow$ „∩" — SKICA: $\Rightarrow$

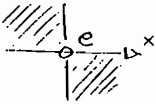


Glavni dio:

1° $E_x: x > 0$

2° Nesimetrično definiciono područje

3° Znak: Zavisi od $1-\ln x \Rightarrow 1-\ln x \Rightarrow x = e$



4° Nule: $x = e$

5° Asimptote:

$x = 0_+,\ y \to +\infty$ V.AS.

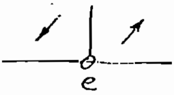
$y = 0_-,\ x \to +\infty$ H.AS.

6° Monotonost:

$2\ln x - 3 \lessgtr 0$ KT: $x = e^{3/2} > e$

za $x > e^{3/2}$ – raste

za $x < e^{3/2}$, $x > 0$ – opada



Lokalni minimum: $y'(e^{3/2}) = 0$.

7° Extremi:

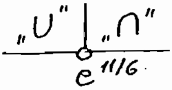
Lokalni: $y' = 0$ na $E'_x \Rightarrow$ ST: $e^{3/2}$

$y(e^{3/2}) = -\frac{1}{2e^3}$

8° Fleksija:

$y'' \gtrless 0$ nad E''_x ; $11 - 6\ln x \gtrless 0$ KT.: $e^{11/6} > e^{3/2} > e$

$\Rightarrow$

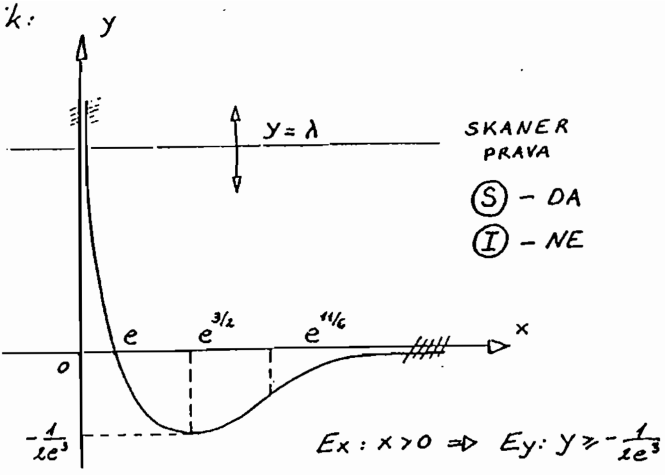


9° Prevoji:

$y'' = 0$ na E''_x, $x_{prev} = e^{11/6}$, $y_{prev} = \frac{1-\frac{11}{6}}{e^{11/3}} = -\frac{5}{6e^{11/3}}$.

Prev. t. $\left(e^{11/6}, -\frac{5}{6e^{11/3}}\right)$

10° Grafik:



* jedno rješenje : $y \in [0, \infty) \cup \{-\frac{1}{2e^3}\}$
* dva rješenja : $-1/2e^3 < \lambda < 0$
* nema rješenja : $\lambda < -\frac{1}{2e^3}$

(Z) Dat je niz f-ja : $f_1(x) = x$

$f_2(x) = \frac{1}{x}$

$f_3(x) = x^2$

$f_4(x) = e^x$

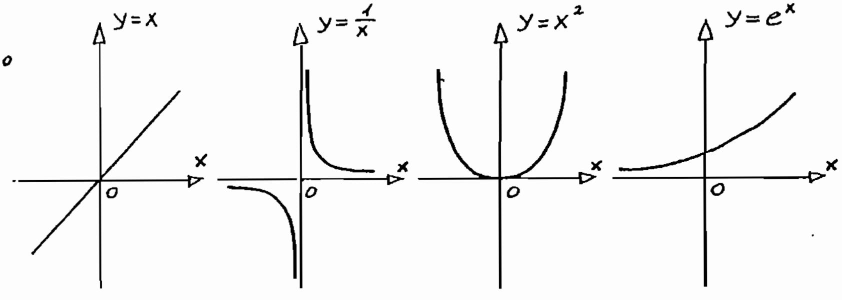
1° Skicirati pojedinačno grafike navedenih f-ja !

2° Ispitati i nacrtati grafik f-je dat sa :

$$y = 1 - f_1(x) \cdot f_3\left(f_4\left(f_2\left(-\frac{1}{x}\right)\right)\right)$$

RJ:

1°



$$2° \quad y = 1 - f_1(x) \cdot f_3\left(f_4\left(f_2\left(-\frac{1}{x}\right)\right)\right) = 1 - x\left(f_4\left(f_2\left(-\frac{1}{x}\right)\right)\right)^2 =$$

$$= 1 - x \cdot e^{2 \cdot f_2\left(-\frac{1}{x}\right)} = \underline{1 - xe^{-\frac{2}{x}}}.$$

$$y = 1 - xe^{-\frac{2}{x}}$$

$y = 1 - xe^{-\frac{2}{x}}$

1° $D(f(x))$: $x \in (-\infty, 0) \cup (0, +\infty)$ - Definiciono područje je simetrično pa ima smisla ispitivati simetriju.

2° Simetrija:

$y(-x) = 1 + xe^{\frac{2}{x}} \neq -y(x),\ \neq y(x)$ - pa f-ja nije simetrična!

3° Neprekidnost: F-je f_1, f_2, f_3 i f_4 su elementarno neprekidne u definicionom području noše f-je pa je i ona neprekidna!

4° Izvodi:

$$y' = (1 - x\cdot e^{-\frac{2}{x}})' = 0 - (x\cdot e^{-\frac{2}{x}})' = -(x\cdot e^{-\frac{2}{x}})' = -(x'\cdot e^{-\frac{2}{x}} + (e^{-\frac{2}{x}})'\cdot x) =$$

$$= -\left(e^{-\frac{2}{x}} + x\cdot e^{-\frac{2}{x}}\left(-\frac{2}{x}\right)'\right) = -\left(e^{-\frac{2}{x}} + xe^{-\frac{2}{x}}\left(-\frac{2'x - x'2}{x^2}\right)\right) =$$

$$= -\left(e^{-\frac{2}{x}} + xe^{-\frac{2}{x}}\left(\frac{2}{x^2}\right)\right) = -\left(e^{-\frac{2}{x}}\left(1 + \frac{2}{x}\right)\right) = -e^{-\frac{2}{x}}\frac{x+2}{x}$$

$y' = -\dfrac{x+2}{x\cdot e^{\frac{2}{x}}}$ a) $D(f'(x)) \equiv D(f(x))$ ili $E_x' \equiv E_x$

b) ST. I.: $-(x+2) = 0 \Rightarrow \underline{x = -2}$ - mogući ekstrem.

Ispitajmo prvi izvod:

$y' = \dfrac{-(x+2)}{x\cdot e^{\frac{2}{x}}}$

$e^{\frac{2}{x}} > 0$ - da vidimo da li je $e^{\frac{2}{x}} > 0\ \forall x \in R \setminus \{0\}$

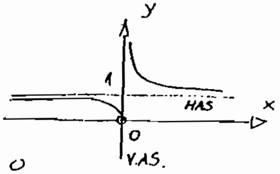
$y = e^{\frac{2}{x}} \Rightarrow$

$\lim\limits_{x \to -\infty} e^{\frac{2}{x}} = e^{\frac{2}{-\infty}} = e^{0-} = \underline{e^0 = 1}$

$\lim\limits_{x \to +\infty} e^{\frac{2}{x}} = e^{\frac{2}{+\infty}} = e^{0+} = \underline{e^0 = 1}$

$\lim\limits_{x \to 0-} e^{\frac{2}{x}} = e^{\frac{2}{0-}} = e^{-\infty} = \frac{1}{e^{\infty}} = \underline{\frac{1}{\infty} = 0}$

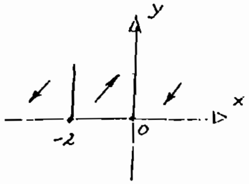
$\lim\limits_{x \to 0+} e^{\frac{2}{x}} = \underline{e^{+\infty} = +\infty}$



Znak naše f-je (izvodne) zavisit će samo od faktora: $-(x+2)$ i x

	$-\infty$ … -2	-2 … 0	0 … $+\infty$
$-x-2$	+	0 −	−
x	−	−	0 +
	−	+	−
	↘	↗	↘

$\Longrightarrow$



$$y''=\left(-\frac{x+2}{x\cdot e^{2/x}}\right)'=-\frac{1\cdot x\cdot e^{2/x}-\left(x'\cdot e^{2/x}+(e^{2/x})'\cdot x\right)\cdot(x+2)}{x^2e^{4/x}}=$$

$$=-\frac{xe^{2/x}-\left(e^{2/x}+e^{2/x}\left(-\frac{2}{x^2}\right)\cdot x\right)\cdot(x+2)}{x^2\cdot e^{4/x}}=-\left[\frac{xe^{2/x}-\left(e^{2/x}-\frac{2}{x}\cdot e^{2/x}\right)(x+2)}{x^2\cdot e^{4/x}}\right]=$$

$$=-\left[\frac{xe^{2/x}-e^{2/x}\left(1-\frac{2}{x}\right)(x+2)}{x^2e^{4/x}}\right]=-\left[\frac{xe^{4/x}-e^{\frac{2}{x}}\left(x+2-2-\frac{4}{x}\right)}{x^2e^{4/x}}\right]=$$

$$=\frac{-\left[xe^{2/x}-e^{2/x}\cdot\left(x-\frac{4}{x}\right)\right]}{x^2e^{4/x}}=\frac{-\left[e^{2/x}\left(x-x+\frac{4}{x}\right)\right]}{x^2e^{4/x}}=\frac{-\frac{4}{x}}{x^2e^{2/x}}=-\frac{4}{x^3e^{\frac{2}{x}}}$$

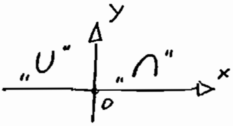
$y''_{(x)}=-\dfrac{4}{x^3e^{\frac{2}{x}}}$. a) $D(f''_{(x)})$: $x\neq0 \Rightarrow D(f''_{(x)})\equiv D(f'_{(x)})\equiv D(f_{(x)})$

b) $y''_{(x)}=0 \Rightarrow$ ni za jedno $x\in\mathbb{R}$.

Ispitajmo sada II izvod:

$y''_{(x)}=-\dfrac{4}{\underbrace{x^3e^{2/x}}_{>0}} \Rightarrow$ znak zavisi samo od x^3.

$y''_{(x)}>0$ za $x<0 \Rightarrow$ „∪"
$y''_{(x)}<0$ za $x>0 \Rightarrow$ „∩"



5° Asimptote:

Potrebno je ispitati tačke: $\pm\infty$ i 0.

$$\lim_{x\to+\infty}\left(1-xe^{-\frac{2}{x}}\right)=\lim_{x\to+\infty}\left(1-\frac{x}{e^{2/x}}\right)=1-\frac{+\infty}{\underset{\longrightarrow 1}{e^0}}=\underline{-\infty}.$$

$$\lim_{x\to-\infty}\left(1-xe^{-\frac{2}{x}}\right)=\lim_{x\to-\infty}\left(1-\frac{x}{e^{2/x}}\right)=1-\frac{-\infty}{\underset{\longrightarrow 1}{e^0}}=\underline{+\infty}.$$

$$\lim_{x\to0-}\left(1-xe^{-\frac{2}{x}}\right)=\lim_{x\to0-}\left(1-\frac{x}{e^{2/x}}\right)=1-\frac{x}{e^{-\infty}}=1-x\cdot e^{\infty}=\underline{\underline{1-0\cdot\infty}}\Rightarrow$$

$0\cdot\infty$ – je neodređeni tip pa moramo imati regularnu transformaciju.

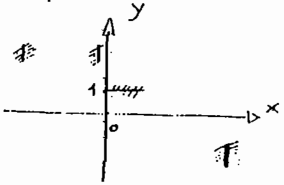
1. $\frac{0}{0}$
2. $\frac{\infty}{\infty}$
3. $0\cdot\infty$
4. $\infty-\infty$
5. 1^{∞}
6. 0^0

Ovi oblici nam govore kada trebamo „ručno" prestati raditi limes, a kada trebamo upotrijebiti transformaciju!

$$=\lim_{x\to0-}\left(1-\frac{x}{e^{2/x}}\right)=1-\lim_{x\to0-}\frac{x}{e^{2/x}}=\left|\begin{array}{l}x=-\frac{1}{t}\\ t\to+\infty\end{array}\right|=1-\lim_{t\to+\infty}\frac{-\frac{1}{t}}{e^{-2\cdot t}}=$$

$$= \left| t \equiv x \right| = 1 - \lim_{t\to+\infty} - \frac{e^{2x}}{x} = 1 + \lim_{x\to+\infty} \frac{1}{\underbrace{\frac{x}{e^{2x}}}_{\to 0}} = 1 + \infty = \underline{+\infty}.$$

$*\ \frac{x}{e^{2x}} \to 0,\ x \to +\infty$ – ovo je standardni Limes.



$$\lim_{x\to 0_+} \left(1 - x e^{-\frac{2}{x}}\right) = 1 - \frac{x}{e^{2/x}} = 1 - \frac{0}{e^{+\infty}} = \underline{1}.$$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \left(\frac{1}{x} - \frac{1}{e^{2/x}}\right) = 0 - \lim_{x\to+\infty} \frac{1}{e^{2/x}} = \underline{-1}.$$

$$\lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty} \left(\frac{1}{x} - \frac{1}{e^{2/x}}\right) = 0 - \lim_{x\to-\infty} \frac{1}{e^{0}} = \underline{-1}$$

$$\lim_{x\to+\infty} (f(x) - ax) = \lim_{x\to+\infty} (f(x) + x) = 1 + \lim_{x\to+\infty} \left(x - x e^{-\frac{2}{x}}\right) =$$

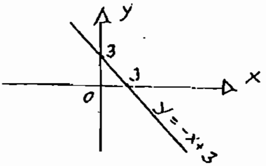
$$= 1 + \lim_{x\to+\infty} \left(x - \frac{x}{e^{2/x}}\right) \overset{\infty-\infty}{=} 1 + \lim_{x\to+\infty} \left(\frac{x e^{2/x} - x}{e^{2/x}}\right) = 1 + \lim_{x\to+\infty} \frac{x(e^{2/x}-1)}{e^{2/x}} \overset{\infty\cdot 0}{=}$$

$$= \left| \text{Z.R.T.} \right| = \left| \frac{2}{x} = t \Rightarrow x = \frac{2}{t},\ t\to 0 \right| = 1 + \lim_{t\to 0} \frac{\frac{2}{t}(e^{t}-1)}{e^{t}} = 1 + \underbrace{\frac{e^{t}-1}{t}}_{\to 1} \cdot \underbrace{\frac{2}{e^{t}}}_{\to 2} = \underline{3}.$$

$$\lim_{x\to-\infty} \left(1 - \frac{x}{e^{2/x}} + x\right) = 1 + \lim_{x\to-\infty} \left(x - \frac{x}{e^{2/x}}\right) = 1 + \lim_{x\to-\infty} \left(\frac{e^{2/x}x - x}{e^{2/x}}\right) =$$

$$= 1 + \lim_{x\to-\infty} \frac{x(e^{\frac{2}{x}}-1)}{e^{2/x}} = \left| \frac{2}{x} = t \Rightarrow x = \frac{2}{t},\ t\to 0_- \right| =$$

$$= 1 + \lim_{t\to 0} \left(\underbrace{\frac{e^{t}-1}{t}}_{\to 1} \cdot \underbrace{\frac{2}{e^{t}}}_{\to 2}\right) = 1 + 2 = 3$$

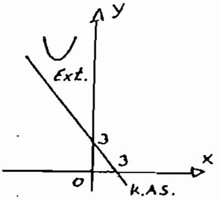


$\Longrightarrow$ KAS: $y = ax + b \Rightarrow \boxed{y = -x + 3}$ K.A.S.

6° Extremi:

kandidat je točka $x = -2$.

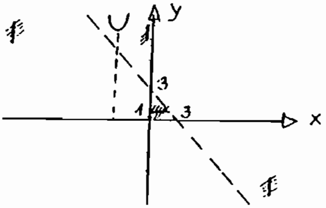
$$y''(x) = -\frac{4}{x^3 e^{2/x}}$$



$$y''(-2) = \frac{-4}{-8 \cdot e^{-1}} = \frac{1}{2} e = \frac{e}{2} > 0 \Rightarrow \underline{\text{minimum}}$$

$$y(-2) = 1 - (-2) e^{-\frac{2}{-2}} = 1 - (-2) e^{1} = 1 + 2 \cdot e \cong 6{,}44. \quad \underline{\text{Ext.1.}(-2,\ 6{,}44)}$$

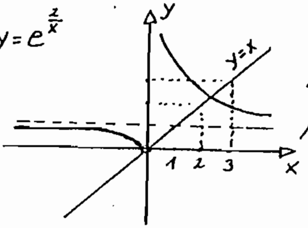
7° Možemo nacrtati sljedeću skicu:



Sa skice vidimo da bi trebali provjeriti nule f-je, i ispitati da li kriva sječe asimptotu!

1) $y = 1 - xe^{-\frac{2}{x}}$ $\quad y = 0$ za $1 - xe^{-\frac{2}{x}} = 0 \Rightarrow xe^{-\frac{2}{x}} = 1 \Rightarrow \frac{x}{e^{2/x}} = 1 \Rightarrow$

$y = e^{\frac{2}{x}}$



$\Rightarrow x = e^{2/x}$. Ovako je nemoguće naći nulu, zato ćemo probati grafičkim putem!

Vidimo da imamo jedno rješenje. Pretpostavimo da je (baš kao na slici) između tačaka 2 i 3.

$x = e^{\frac{2}{x}}$

za $x = 2 \Rightarrow 2 = e^{1} \Rightarrow 2 = e^{1} \Rightarrow \boxed{2 < e}$ $\quad\Rightarrow$ nula jeste

za $x = 3 \Rightarrow 3 = e^{2/3} \Rightarrow 3 = \sqrt[3]{e^2} \Rightarrow \boxed{3 > 1{,}947}$ $\quad$ između 2 i 3.

Naša pretpostavka je tačna!

2)

$1 - xe^{-\frac{2}{x}} = -x + 3 \Rightarrow 1 - 3 = xe^{-\frac{2}{x}} - x \Rightarrow -2 = x(e^{-\frac{2}{x}} - 1) \Rightarrow$

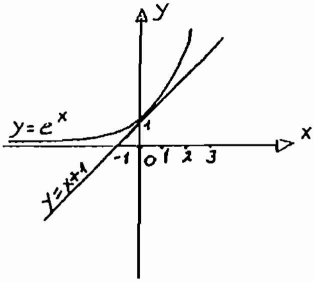
$\Rightarrow x(e^{-\frac{2}{x}} - 1) = -2 \Rightarrow e^{-\frac{2}{x}} = -\frac{2}{x} + 1 \Rightarrow \frac{1}{x} = u \Rightarrow e^{-u \cdot 2} = -2u + 1 \Rightarrow$

$\Rightarrow -u = x \Rightarrow e^{2x} = 2x + 1 \Rightarrow 2x = t = x \Rightarrow \underline{e^{x} = x + 1}$.

za $x = -1 \Rightarrow \frac{1}{e} = 0 \Rightarrow \frac{1}{e} > 0$

za $x = 1 \Rightarrow e = 2 \Rightarrow e > 2$

za $x = 0 \Rightarrow 1 = 1 \Rightarrow$ (T) – međutim f-ja nije definisana u tački $x = 0$.



– Sa slike također vidimo da je presjek u $x = 0$ jer obe f-je prolaze kroz tačku $(0, 1)$.

Možemo još provjeriti ugao ulaska f-je, izlaska f-je iz tačke 1.

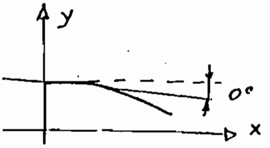
$$\lim_{x \to 0_+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0_+} \frac{1 - xe^{-\frac{2}{x}} - 1}{x - 0} = \lim_{x \to 0_+} \frac{-x \cdot e^{-\frac{2}{x}}}{x} = \lim_{x \to 0_+} \frac{-x \cdot 1}{x e^{2/x}} =$$

$= \lim\limits_{x\to 0_+} \dfrac{-1}{e^{2/x}} = -\dfrac{1}{e^{+\infty}} = -\dfrac{1}{+\infty} = 0_-$

Kako je: $\lim\limits_{x\to 0_+} \dfrac{f(x)-f(0)}{x-0} = tg\,\alpha \Rightarrow 0_- = tg\,\alpha \Rightarrow \alpha = arctg\, 0 \Rightarrow$

$\Rightarrow \underline{\alpha = 0}.$

SKICA:



* Ugao izlaska je provjeren preko definicije izvoda f. preko formule: $\lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a}$.

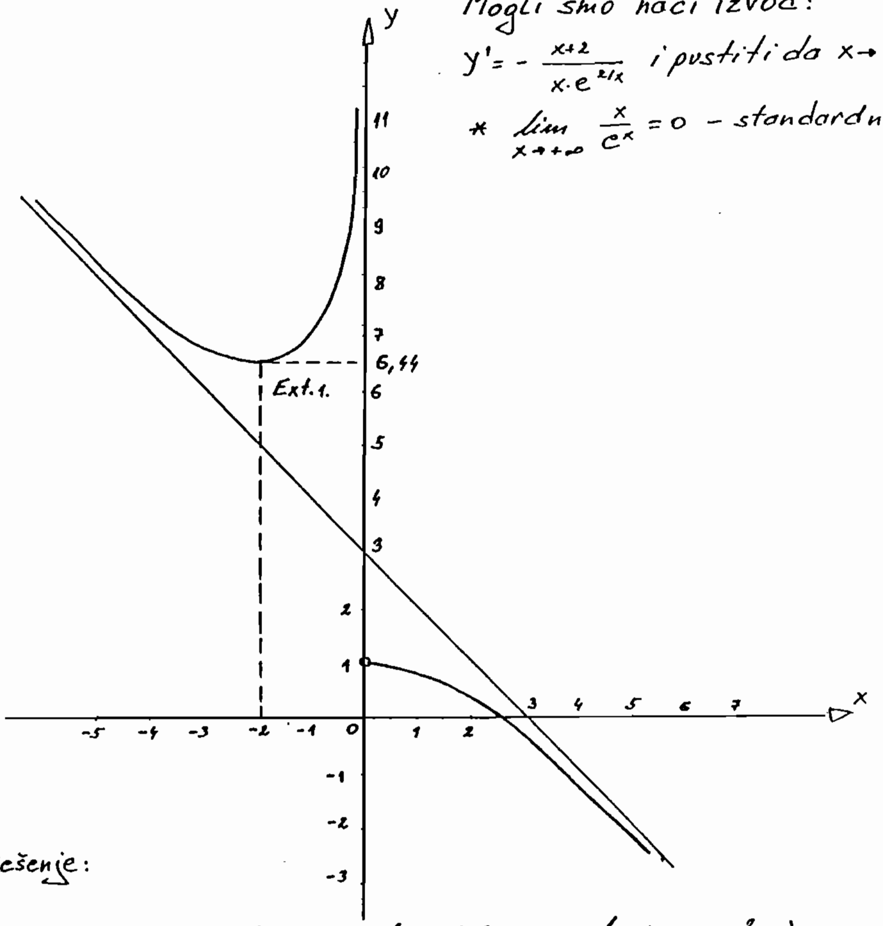
Ovo je ustvari traženje izvoda u tački a.

Mogli smo naći izvod:

$y' = -\dfrac{x+2}{x\cdot e^{2/x}}$ i pustiti da $x \to 0_+$.

* $\lim\limits_{x\to +\infty} \dfrac{x}{e^x} = 0$ – standardni

8° Grafik:



Rješenje:

*

$\lim\limits_{x\to 0_+} y' = \lim\limits_{x\to 0_+} -\dfrac{x+2}{xe^{2/x}} = -\lim\limits_{x\to 0_+} \dfrac{x+2}{\underbrace{xe^{2/x}}_{0\cdot\infty}} = -\lim\limits_{x\to 0_+}\left(\dfrac{x}{xe^{2/x}} + \dfrac{2}{xe^{2/x}}\right) =$

$= -\lim\limits_{x\to 0_+}\left(\dfrac{1}{e^{2/x}}\nearrow^{0} + \dfrac{2}{\underbrace{x\cdot e^{2/x}}_{0\cdot\infty}}\right) = -\lim\limits_{x\to 0_+} \dfrac{2}{x\cdot e^{2/x}} = \left| \begin{matrix} t = \frac{2}{x} \\ t \to +\infty \end{matrix} \Rightarrow x = \frac{2}{t} \right| =$

$= -\lim \dfrac{2}{\frac{2}{t}\cdot e^{t}} = -\lim \dfrac{x}{e^x} = \underline{0_-}$

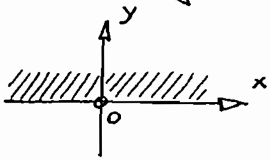
(2) $y = \frac{x^2+2}{x}$

1° $D(f(x)): \forall x \neq 0 \Rightarrow x \in (-\infty, 0) \cup (0, +\infty)$

2° Simetrija: Ima smisla ispitivati simetriju jer je def. područje simetrično.

$y(-x) = \frac{x^2+2}{-x} = -\frac{x^2+2}{x} = -y(x)$

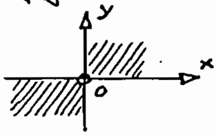
pa je f-ja simetrična u odnosu na koordinatni početak.



3° Neprekidnost:

kako su f-je: x^2, 2, x elementarno neprekidne to je i kompozicija ovih f-ja, tj. naša f-ja neprekidna.

4° Znak f-je: $\left.\begin{array}{l} y>0 \text{ za } x>0 \\ y<0 \text{ za } x<0 \end{array}\right\} \Rightarrow$



5° Izvodi:

$y'(x) = \left(\frac{x^2+2}{x}\right)' = \frac{2x \cdot x - (x^2+2) \cdot 1}{x^2} = \frac{2x^2 - x^2 - 2}{x^2} = \frac{x^2-2}{x^2}$

$y''(x) = \left(\frac{x^2-2}{x^2}\right)' = \frac{2x \cdot x^2 - 2x(x^2-2)}{x^4} = \frac{2x^3 - 2x^3 + 4x}{x^4} = \frac{4x}{x^4} = \frac{4}{x^3}$

6° Obrada prvog izvoda:

$y'(x) = \frac{x^2-2}{x^2}$

1) $D(f')$: $x \neq 0 \Rightarrow x=0$ ST. I. $\Rightarrow x=0$ nije ST. I jer $f(x)$ u $x=0$ nije definisana.

2) $y'(x) = 0 \Rightarrow x^2 - 2 = 0 \Rightarrow x^2 = 2 \Rightarrow x_1 = -\sqrt{2}$, $x_2 = \sqrt{2}$, i ovo su ST. I tj. mogući ekstremi.

3) $y'(x) = \frac{x^2-2}{x^2} = \frac{(x-\sqrt{2})(x+\sqrt{2})}{x^2}$, x^2 - ne utiče na znak.

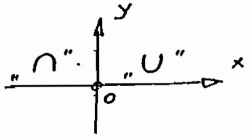
	$-\infty$ … $-\sqrt{2}$	$-\sqrt{2}$ … $\sqrt{2}$	$\sqrt{2}$ … $+\infty$
$x+\sqrt{2}$	−	+	+
$x-\sqrt{2}$	−	−	+
	+	−	+
	↗	↘	↗

monotonija

7° Obrada drugog izvoda;

$y''(x) = \frac{4}{x^3} \Rightarrow x \neq 0 \Rightarrow$ je ST. II. $\Rightarrow x=0$ nije ST. II. jer f-ja nije definisana u toj tački.

$y''(x) = \frac{4}{x^3}$ $\quad >0$ za $x>0 \Rightarrow$ „U"
$\quad\quad\quad\quad <0$ za $x<0 \Rightarrow$ „∩"



8° Limesi i asimptote :

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x^2+2}{x} = \lim_{x\to+\infty} \frac{1+\frac{2}{x^2}}{\frac{1}{x}} = \frac{1}{0_+} = \underline{+\infty}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2+2}{x} = \lim_{x\to-\infty} \frac{1+\frac{2}{x^2}}{\frac{1}{x}} = \frac{1}{0_-} = \underline{-\infty}.$$

$$\lim_{x\to0_-} \left(\frac{x^2+2}{x}\right) = \lim_{x\to0_-} \left(\frac{x^2}{x} + \frac{2}{x}\right) = \lim_{x\to0_-} \left(x+\frac{2}{x}\right) = \underline{-\infty}.$$

$$\lim_{x\to0_+} \left(\frac{x^2+2}{x}\right) = \lim_{x\to0_+} \left(x+\frac{2}{x}\right) = \underline{+\infty}.$$

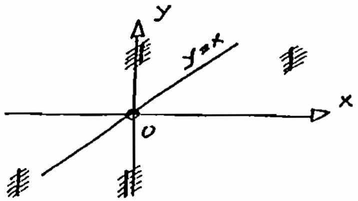
$$\lim_{x\to+\infty} \left(\frac{f(x)}{x}\right) = \lim_{x\to+\infty} \left(\frac{x^2-2}{x^2}\right) = 1-\frac{2}{x^2} = \underline{1}.$$

$$\lim_{x\to-\infty} \left(\frac{f(x)}{x}\right) = \lim_{x\to-\infty} \left(\frac{x^2}{x^2}-\frac{2}{x^2}\right) = 1-0 = \underline{1}.$$

$$\lim_{x\to+\infty} (f(x)-x) = \lim_{x\to+\infty} \left(\frac{x^2-2}{x}-x\right) = \frac{x^2-2-x^2}{x} = \lim_{x\to+\infty} \left(-\frac{2}{x}\right) = \underline{0_-}.$$

$$\lim_{x\to-\infty} (f(x)-x) = \lim_{x\to-\infty} \left(\frac{x^2-2}{x}-x\right) = \lim_{x\to-\infty} \left(-\frac{2}{x}\right) = \underline{0_+}.$$

$\Rightarrow a_{1,2} = 1, \; b_{1,2} = 0 \Rightarrow \; y=x$ - dvije kose AS.



9° Extremi :

$y = \frac{x^2+2}{x} \quad \Rightarrow$ ST. I : $\; x_1 = -\sqrt{2}$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad x_2 = \sqrt{2}$

$y' = \frac{x^2-2}{x^2}$

$y''(x) = \frac{4}{x^3} \quad \Rightarrow \quad y''(-\sqrt{2}) = \frac{4}{(-\sqrt{2})^3} = \frac{4}{-2\sqrt{2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2} < 0$

$x = -\sqrt{2}$ je maximum.

$y(-\sqrt{2}) = \frac{2+2}{-\sqrt{2}} = -\frac{4}{\sqrt{2}} = -2\sqrt{2} = -2{,}82.$

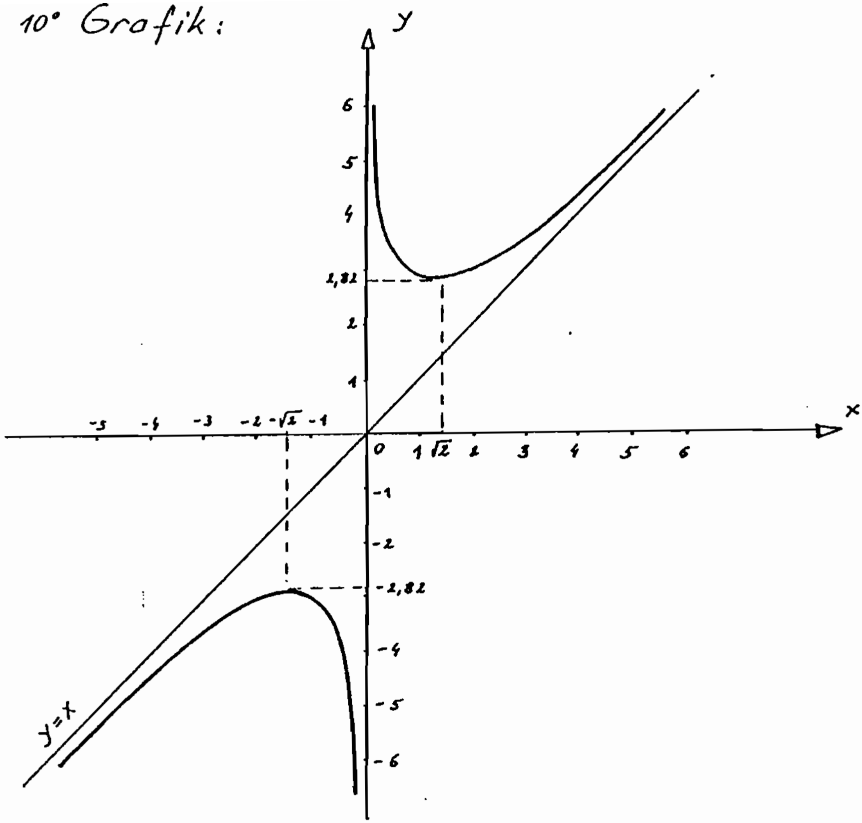
$y''(\sqrt{2}) = \frac{4}{2\sqrt{2}} = \sqrt{2} > 0 \Rightarrow$ minimum.

$y(\sqrt{2}) = 2\sqrt{2} = 2{,}82.$

Ext. 1. (−1,41 , −2,82)

Ext. 2. (1,41 , 2,82)

10° Grafik:



(2) Ispitati tok i nacrtati grafik f-je:

$y=\sqrt[3]{x^3-x^2}$.

RJ:

1° $D(f(x))$: $\forall x \in \mathbb{R}$ SKICA:

-treći korijen nema ograničenja.

2° Nule:

$y=0$ za $\sqrt[3]{x^3-x^2}=0 \Rightarrow x^3-x^2=0 \Rightarrow x^2(x-1)=0 \Rightarrow$

$\Rightarrow x_{1,2}=0, \quad x_3=1.$

3° Simetričnost: (def. područje je simetrično):

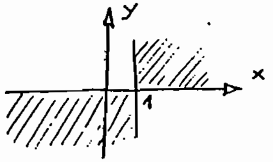
$y(-x)=\sqrt[3]{-x^3-x^2} \neq -y(x), \neq y(x)$ pa f-ja nije simetrična!

4° Znak f-je:

$y = \sqrt[3]{x^3 - x^2} = \sqrt[3]{x^2(x-1)}$. Znak zavisi samo od faktora $(x-1)$.

	$-\infty$	1	$+\infty$
$x-1$	$-$	ϕ	$+$
	$-$		$+$

SKICA $\Rightarrow$



5° Neprekidnost: kako je svaki faktor u f-ji neprekidan (elementarno) to je i naša f-ja neprekidna.

6° Izvodi:

$y = \sqrt[3]{x^3 - x^2}$

$$y'_{(x)} = \left((x^3-x^2)^{\frac{1}{3}}\right)' = \frac{1}{3}(x^3-x^2)^{\frac{1}{3}-\frac{3}{3}} \cdot (3x^2-2x) = \frac{3x^2-2x}{3\cdot\sqrt[3]{(x^3-x^2)^2}} =$$

$$= \frac{x(3x-2)}{3\sqrt[3]{(x^3-x^2)^2}} = \frac{x(3x-2)}{3\sqrt[3]{(x^2(x-1))^2}} = \frac{x(3x-2)}{3\cdot\sqrt[3]{x^4(x-1)^2}} = \frac{3x-2}{3\sqrt[3]{x(x-1)^2}}$$

7° $$y''_{(x)} = \left(\frac{3x^2-2x}{3\sqrt[3]{(x^3-x^2)^2}}\right)' = \frac{1}{3}\left(\frac{3x^2-2x}{\sqrt[3]{(x^3-x^2)^2}}\right)' =$$

* $$\left(\sqrt[3]{(x^3-x^2)^2}\right)' = \left((x^3-x^2)^{2/3}\right)' = \frac{2}{3}(x^3-x^2)^{-\frac{1}{3}}\cdot(3x^2-2x) =$$

$$= \frac{2(3x^2-2x)}{3\sqrt[3]{x^3-x^2}}$$

* $(3x^2-2x)' = 6x-2$

$$= \frac{(6x-2)\cdot\sqrt[3]{(x^3-x^2)^2} - \frac{2}{3}\frac{(3x^2-2x)}{\sqrt[3]{x^3-x^2}}(3x^2-2x)}{3\cdot\sqrt[3]{(x^3-x^2)^4}} =$$

$$= \frac{(6x-2)\sqrt[3]{(x^3-x^2)^2} - \frac{2}{3}\cdot\frac{(3x^2-2x)^2}{\sqrt[3]{x^3-x^2}}}{3\cdot\sqrt[3]{(x^3-x^2)^4}} = \frac{3\cdot(6x-2)\sqrt[3]{(x^3-x^2)^3} - 2(3x^2-2x)^2}{9\cdot\sqrt[3]{(x^3-x^2)^5}} =$$

$$= \frac{3x^2(6x^2-6x-2x+2) - 2x^2(9x^2-12x+4)}{9\sqrt[3]{(x^3-x^2)^5}} = \frac{x^2(18x^2-24x+6-18x^2+24x-8)}{9\sqrt[3]{(x^3-x^2)^5}} =$$

$$= \frac{-2x^2}{9\sqrt[3]{(x^3-x^2)^5}} = \frac{-2x^2}{9x^2\cdot\sqrt[3]{x^4(x-1)^5}} = -\frac{2}{9\sqrt[3]{x^4(x-1)^5}} =$$

$$= \frac{2}{9}\cdot\frac{1}{\sqrt[3]{x^4(1-x)^5}} = \frac{2}{9\sqrt[3]{x^4(1-x)^5}}$$

7° Ispitivanje prvog izvoda:

$$y'(x) = \frac{3x-2}{3\cdot\sqrt[3]{x(x-1)^2}}$$

a) $D(f'(x)): \left.\begin{matrix} x\neq 0 \\ x\neq 1 \end{matrix}\right\}$ ST. I.

b) $y'(x)=0$ za $3x-2=0 \Rightarrow x=\frac{2}{3}$ } ST. I.

Faktori koji mijenjaju znak su: x i $3x-2$.

	$-\infty$ … 0	0 … $\frac{2}{3}$	$\frac{2}{3}$ … $+\infty$
x	$-$	$+$	$+$
$3x-2$	$-$	$-$	$+$
	$+$	$-$	$+$
	↗	↘	↗

8° Ispitivanje drugog izvoda:

$$y''(x) = -\frac{2}{9}\,\frac{1}{\sqrt[3]{x^4(1-x)^5}}$$

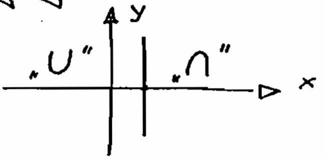
a) $D(f''(x)): \left.\begin{matrix} x\neq 0 \\ x\neq 1 \end{matrix}\right\}$ ST. II

b) Nula nema!

$(1-x)$ - je jedini faktor koji mijenja znak: $1-x=0,\ x=1$.

$y''(x) > 0$ za $x<1 \Rightarrow$ „U"

$y''(x) < 0$ za $x>1 \Rightarrow$ „∩"



$x=1$ je prevojna tačka

9° Ispitivanje definicionog područja i asimptota:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\sqrt[3]{x^3-x^2} = \sqrt[3]{\lim_{x\to+\infty}(x^3-x^2)} = \lim_{x\to+\infty} x\sqrt[3]{1-\frac{1}{x}} =$$

$$= \lim_{x\to+\infty} x\sqrt[3]{1-\frac{1}{x}} = \underline{+\infty}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x\cdot\underbrace{\sqrt[3]{1-\frac{1}{x}}}_{\to 1} = \underline{-\infty}.$$

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{\sqrt[3]{x^3-x^2}}{x} = \lim_{x\to+\infty}\sqrt[3]{\frac{x^3-x^2}{x^3}} = \sqrt[3]{1-\frac{1}{x}} = \underline{1}.$$

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\sqrt[3]{1-\frac{1}{x}} = \underline{1}.$$

$$\lim_{x\to+\infty}(f(x)-ax) = \lim_{x\to+\infty}(f(x)-x) = \lim_{x\to+\infty}\left(\sqrt[3]{x^3-x^2}-x\right) =$$

$$= \lim_{x\to+\infty} x\left(\sqrt{1-\frac{1}{x}}-1\right) = \infty\cdot 0 = |\text{Z.R.T.}| = |\text{zahtjeva regularnu}$$

$$\text{transformaciju}| = \lim_{x\to+\infty}\left(\sqrt[3]{x^3-x^2}-x\right) = \left|\sqrt[3]{a}-\sqrt[3]{b} = \frac{a-b}{\sqrt[3]{a^2}+\sqrt[3]{ab}+\sqrt[3]{b^2}}\right| =$$

$$= \lim_{x\to+\infty}\left(\sqrt[3]{x^3-x^2}-\sqrt[3]{x^3}\right) = \lim_{x\to+\infty}\frac{x^3-x^2-x^3}{\sqrt[3]{(x^3-x^2)^2}+\sqrt[3]{(x^3-x^2)\cdot x^3}+\sqrt[3]{x^6}} =$$

$$= \lim_{x\to+\infty}\frac{-x^2}{x^2\Big(\underbrace{\sqrt[3]{(1-\frac{1}{x})^2}}_{1}+\underbrace{\sqrt[3]{1-\frac{1}{x}}}_{1}+\underbrace{1}_{1}\Big)} = -\frac{1}{3}.$$

$$\lim_{x\to-\infty}(f(x)-ax) = \lim_{x\to-\infty}\frac{-x^2}{x^2\left(\sqrt[3]{(1-\frac{1}{x})^2}+\sqrt[3]{1-\frac{1}{x}}+1\right)} = -\frac{1}{3}.$$

Dobili smo dvije kose asimptote u formuli:

$y = ax+b \Rightarrow y = x-\frac{1}{3}$ K.AS.

Limes smo mogli uraditi i ovako:

$$\lim_{x\to-\infty}(y-ax) = \lim_{x\to-\infty}\left(\sqrt[3]{x^3-x^2}-x\right) = \left|\begin{matrix} x=-\frac{1}{t} \\ t\to 0_+\end{matrix}\right| = \left| t \equiv x\right| =$$

$$= \lim_{x\to 0_+}\frac{1+\sqrt[3]{-1-t}}{t} = \lim_{x\to 0_+}\frac{1+\sqrt[3]{-1-x}}{x} = \lim_{x\to 0_+}\frac{1-\sqrt[3]{1+x}}{x} \stackrel{\frac{0}{0}}{=}$$

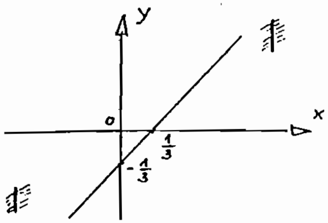
$$= \lim_{x\to 0_+} -\frac{\sqrt[3]{1+x}-1}{x} = -\lim_{x\to 0_+}\frac{\sqrt[3]{1+x}-1}{x} = \left| e^{\ln f(x)} = f(x)\right| =$$

$$= -\lim_{x\to 0_+}\frac{e^{\ln\sqrt[3]{1+x}}-1}{x} = -\lim_{x\to 0_+}\frac{e^{\ln(1+x)^{\frac{1}{3}}}-1}{x} = \left|\frac{e^{f(x)}-1}{f(x)} \xrightarrow{f(x)\to 0} 1\right| =$$

$$= -\lim_{x\to 0_+}\underbrace{\frac{e^{\ln(1+x)^{\frac{1}{3}}}-1}{\ln(1+x)^{\frac{1}{3}}}}_{\to 1}\cdot\frac{\ln(1+x)^{\frac{1}{3}}}{x} = -\lim_{x\to 0_+}\frac{1}{3}\cdot\frac{\ln(1+x)}{x} =$$

$$= -\frac{1}{3}\lim_{x\to 0_+}\underbrace{\frac{\ln(1+x)}{x}}_{\to 1} = -\frac{1}{3}.$$

SKICA:



10° Extremi:

kandidati su: $x=\frac{2}{3}$, $x=0$, $\underline{x=1}$ ⟶ otpada - to je prevoj.

$$y''\left(\tfrac{2}{3}\right) = \frac{2}{9}\cdot\frac{1}{\sqrt[3]{\left(\frac{2}{3}\right)^4\underbrace{\left(1-\frac{2}{3}\right)^5}_{>0}}} > 0 \quad \text{minimum.}$$

Ext. 1. $\left(\frac{2}{3}, -0{,}529\right)$

$$y\left(\tfrac{2}{3}\right) = \sqrt[3]{\left(\tfrac{2}{3}\right)^3-\left(\tfrac{2}{3}\right)^2} = -\frac{\sqrt[3]{4}}{3} = -0{,}529.$$

$y'(0)$ - ne postoji, a f-ja je definisana u tački nula.

- f-ja raste od $-\infty$ do 0, a od 0 do $+\infty$ opada.
- f-ja ne prelazi na ⊕ dio y ose, a ima dodir u tački $x=0$.

Sve ovo nam govori da je $x=0$ vjerovatno šiljak.

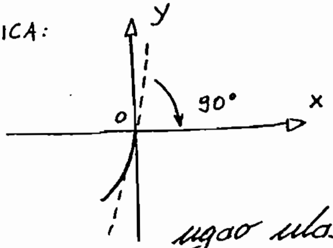
11° Provjera postojanja šiljka u $x=0$.

$$\lim_{x\to 0_-} y' = \lim_{x\to 0_-} \frac{3x-2}{3\cdot\sqrt[3]{x(x-1)^2}} = \frac{-2}{0_-} = -\frac{2}{0_-} = -(-\infty) = \underline{+\infty}.$$

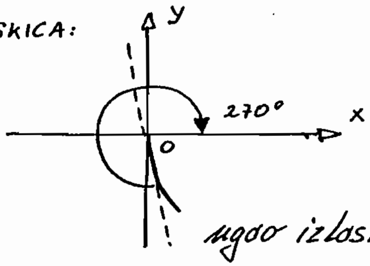
$tg\,\alpha = +\infty \Rightarrow \alpha = arctg(+\infty) \Rightarrow \underline{\alpha \cong 90°}$ - ugao ulaska.

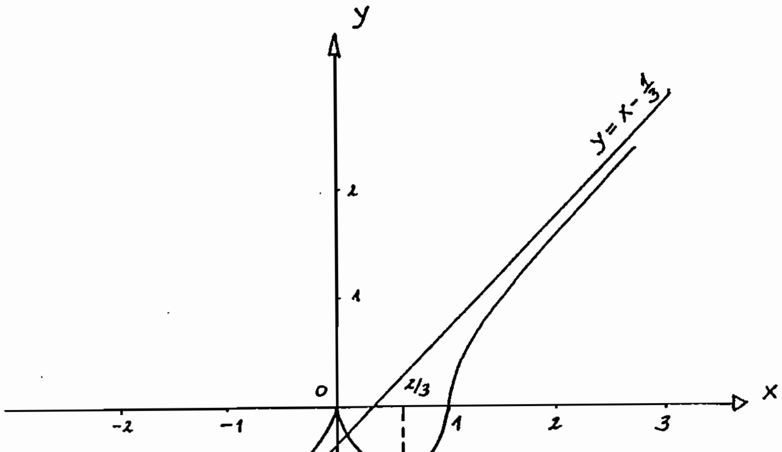
$$\lim_{x\to 0_+} y' = \lim_{x\to 0_+} \frac{-2}{0_+} = -\infty \Rightarrow \underline{\alpha \cong 270°}$$ - ugao izlaska.

SKICA:



SKICA:





* Moguć je presjek krive i kose asimptote, nije moguć, nego očigledan:

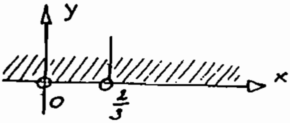
$$\sqrt[3]{x^3-x^2} = x-\frac{1}{3} \quad /^3$$

$$x^3-x^2 = x^3-x^2+\frac{1}{3}x-\frac{1}{27} \Rightarrow$$

$$x=\frac{1}{9}.$$

(Z) Nacrtati grafik f-je : $y = \frac{(x-1)e^{\frac{1}{x}}}{3x-2}$.

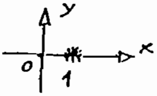
1° $O(f)$: $x \neq 0 \wedge x \neq \frac{2}{3} \Rightarrow$



2° Simetrija:

$y(-x) = \frac{(-x-1)e^{-\frac{1}{x}}}{-3x-2} \Rightarrow \neq y(x), \neq -y(x)$ pa f-ja nije simetrična.

3° Nule f-je: $x=1 \Rightarrow y=0$

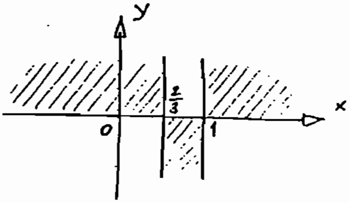


$y(0)$ - ne može - jer f-ja nije definisana u $x=0$

4° Znak f-je: $e^{\frac{1}{x}} > 0 \;\forall x \in \mathbb{R},\; x \neq 0$ - na znak utiču faktori: $(x-1)$, $(3x-2)$.

	$-\infty$ … $\frac{2}{3}$	$\frac{2}{3}$ … 1	1 … $+\infty$
$(x-1)$	−	−	+
$(3x-2)$	−	+	+
	+	−	+

$\Rightarrow$



5° Izvodi:

$$y = \frac{(x-1)e^{1/x}}{(3x-2)} = \frac{\overbrace{\left((x-1)e^{1/x}\right)'}^{①}(3x-2) - 3\cdot(x-1)e^{1/x}}{(3x-2)^2} =$$

① $$\left((x-1)e^{1/x}\right)' = (x-1)'e^{1/x} + \left(e^{1/x}\right)'(x-1) = e^{1/x} + e^{1/x}\left(-\frac{1}{x^2}\right)(x-1) =$$

$$= e^{1/x} - e^{1/x}\frac{(x-1)}{x^2} = e^{1/x}\left(1 - \frac{(x-1)}{x^2}\right) = e^{\frac{1}{x}}\left(\frac{x^2-x+1}{x^2}\right) =$$

$$= \frac{e^{1/x}(x^2-x+1)}{x^2}$$

$$= \frac{e^{1/x}\left(\frac{x^2-x+1}{x^2}\right)(3x-2) - 3(x-1)e^{1/x}}{(3x-2)^2} =$$

$$= \frac{e^{1/x}\left(\left(\frac{x^2-x+1}{x^2}\right)(3x-2) - 3(x-1)\right)}{(3x-2)^2} = \frac{e^{1/x}\left(\left(\frac{3x^2-3x+3}{x} - \frac{2x^2-2x+2}{x^2}\right) - 3x+3\right)}{(3x-2)^2} =$$

$$= \frac{e^{1/x}\left(\frac{3x^3-3x^2+3x-2x^2+2x-2}{x^2} - 3x+3\right)}{(3x-2)^2} = \frac{e^{1/x}(-2x^2+5x-2)}{x^2(3x-2)^2}.$$

$$y'(x) = \frac{e^{1/x}(-2x^2+5x-2)}{x^2(3x-2)^2}.$$

$$y''_{(x)} = \frac{\left(e^{1/x}(-2x^2+5x-2)\right)' x^2(3x-2)^2 - \left(x^2(3x-2)^2\right)' e^{1/x}(-2x^2+5x-2)}{\left(x^2(3x-2)^2\right)^2} =$$

$$= \frac{\overbrace{\left(e^{1/x}(-2x^2+5x-2)\right)'}^{①} x^2(3x-2)^2 - \overbrace{\left(x^2(3x-2)^2\right)'}^{②} e^{1/x}(-2x^2+5x-2)}{\left(x^2(3x-2)^2\right)^2} =$$

① $$\left(e^{1/x}(-2x^2+5x-2)\right)' = \left(e^{1/x}\left(-\frac{1}{x^2}\right)(-2x^2+5x-2) + (-4x+5)e^{1/x}\right) =$$

$$= e^{1/x}\left(\frac{2x^2-5x+2}{x^2} + (-4x+5)\right) = e^{1/x}\left(\frac{2x^2-5x+2-4x^3+5x^2}{x^2}\right) =$$

$$= \frac{e^{1/x}(-4x^3+7x^2-5x+2)}{x^2}$$

②

$$\left(x^2(3x-2)^2\right)' = 2x(3x-2)^2 + 2(3x-2)\cdot 3\cdot x^2 = 2x(3x-2)^2 +$$

$$+ 6x^2(3x-2) = 2x(3x-2)(3x-2+3x) = 2x(3x-2)(6x-2).$$

$$y''_{(x)} = \frac{\frac{e^{1/x}(-4x^3+7x^2-5x+2)}{x^2}\cdot x^2(3x-2)^2 - 2x(3x-2)(6x-2)e^{1/x}(-2x^2+5x-2)}{x^4(3x-2)^4} =$$

$$= \frac{e^{1/x}(-4x^3+7x^2-5x+2)(3x-2)^2 - 2x(6x-2)(3x-2)e^{1/x}(-2x^2+5x-2)}{x^4(3x-2)^4} =$$

$$= \frac{e^{\frac{1}{x}}(3x-2)\left[(-4x^3+7x^2-5x+2)(3x-2) - 2x(6x-2)(-2x^2+5x-2)\right]}{x^4(3x-2)^{\not{4}\,3}} =$$

$$= \frac{e^{1/x}\left((-12x^4+21x^3-15x^2+6x+8x^3-14x^2+10x-4) - (12x^2-4x)(-2x^2+5x-2)\right)}{x^4(3x-2)^3} =$$

$$= \frac{e^{1/x}\left(-12x^4+29x^3-29x^2+16x-4-(-24x^4+60x^3-24x^2+8x^3-20x^2+8x)\right)}{x^4(3x-2)^3} =$$

$$= \frac{e^{1/x}(12x^4-39x^3+15x^2+8x-4)}{x^4(3x-2)^3}$$

$$y''_{(x)} = \frac{e^{1/x}(12x^4-39x^3+15x^2+8x-4)}{x^4(3x-2)^3}$$

6° Ispitivanje izvoda:

$$y'(x) = \frac{e^{1/x}(-2x^2+5x-2)}{x^2(3x-2)^2}$$

a) $D(f'_{(x)})$: $x \neq 0$, $3x-2 \neq 0 \Rightarrow x \neq \frac{2}{3}$ } ST.I.

b) $(-2x^2+5x-2) = 0 \,/\cdot(-1) \Rightarrow 2x^2-5x+2=0 \Rightarrow x_{1,2} = \frac{5\pm\sqrt{25-16}}{4} = \frac{5\pm 3}{4} \Rightarrow$

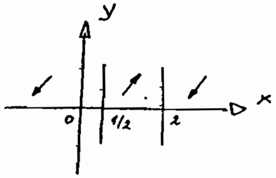
$x_1 = 2$, $x_2 = \frac{1}{2}$ } $\Rightarrow$ $\boxed{-2(x-2)(x-\tfrac{1}{2}) = -2x^2+5x-2}$ (T) $\Rightarrow$

$$y'(x) = \frac{e^{1/x}(-2x^2+5x-2)}{x^2(3x-2)^2} = \frac{e^{1/x}\left(2(2-x)(x-\frac{1}{2})\right)}{x^2(3x-2)^2}.$$

Odavde vidimo da znak y' zavisi samo od faktora: $(2-x)$ i $(x-1/2)$. Formirajmo tablicu:

	$-\infty$ do $1/2$	$1/2$ do 2	2 do $+\infty$
$(x-1/2)$	$-$	$+$	$+$
$(2-x)$	$+$	$+$	$-$
	$-$	$+$	$-$
	↘	↗	↘

SKICA ⟹



Stacionarne tačke su:

$x = \frac{1}{2}$

$x = 2$

$\left.\begin{array}{l} x=0 \\ x=2/3 \end{array}\right\}$ – ovdje f-ja nije definisana pa ove dvije otpadaju.

$$y''(x) = \frac{e^{1/x}(12x^4-39x^3+15x^2+8x-4)}{x^4(3x-2)^3}.$$

a) $D(f''(x))$: $x \neq 0$, $x \neq \frac{2}{3}$

Nisu ST. II jer f-ja u njima nije definisana.

$y''(x)=0$ za $(12x^4-39x^3+15x^2+8x-4)=0$

Ima mnogo mogućih rješenja jer iz: $\pm\frac{4}{12} \Rightarrow \pm\frac{1}{3}, \pm 1, \pm 2, \pm 4, \pm\frac{1}{6}$.

Bilo koji od ovih brojeva može dati nulu.

Ispitajmo ekstreme:

$$y''\left(\tfrac{1}{2}\right) = \frac{\left(12\left(\frac{1}{2}\right)^4 - 39\left(\frac{1}{8}\right) + 12\left(\frac{1}{4}\right) + \frac{8}{2} - 4\right)}{\left(3\cdot\frac{1}{2}-2\right)^3} = \frac{\frac{12}{16} - \frac{39}{8} + \frac{15}{4}}{\frac{3}{2}-\frac{4}{2}} =$$

$$= \frac{\frac{12-78+60}{16}}{-\frac{1}{2}} = \frac{-\frac{6}{16}}{-\frac{1}{2}} = \frac{12}{16} > 0$$

imamo minimum u $x = \frac{1}{2}$.

$$y(1/2) = \frac{\left(\frac{1}{2}-1\right)e^2}{\frac{3}{2}-2} = \frac{-\frac{1}{2}e^2}{-\frac{1}{2}} = e^2$$

Ext. 1. $\left(\frac{1}{2}, 7{,}38\right)$

$$y''(2) = \frac{(12\cdot16 - 39\cdot8 + 15\cdot4 + 8\cdot2 - 4)}{4^3} = \frac{192-312+60+16-4}{4^3} = \frac{-50}{4^3} < 0$$

maximum

$$y(2) = \frac{(2-1)e^{1/2}}{6-2} = \frac{e^{1/2}}{4} + \frac{\sqrt{e}}{4} = 0{,}41.$$

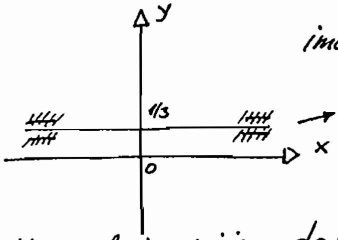
Ext. 2. $(2, 0{,}41)$

7° Ispitivanje definicionog područja:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{(x-1)e^{1/x}}{(3x-2)} = \frac{(1-\frac{1}{x})e^{1/x}}{(3-\frac{2}{x})} = \frac{1}{3}.$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{3} e^{\frac{1}{-\infty}} = \frac{1}{3}\frac{1}{e^{1/\infty}} = \frac{1}{3}.$$

SKICA:



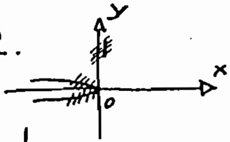
imamo dvije H.AS.

→ još ne znamo da li je f-ja iznad ili ispod H.AS.

Tačke u kojima f-ja nije definisana su:

$x=0$, $x=\frac{2}{3}$

$$\lim_{x\to 0-} \frac{(x-1)e^{1/x}}{(3x-2)} = \frac{1}{2}e^{\frac{1}{0-}} = \frac{1}{2e^{\infty}} = \frac{1}{2\infty} = \underline{0}.$$

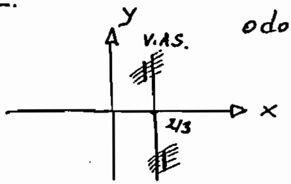


↳ teži nuli ili odozgo ili odozdo.

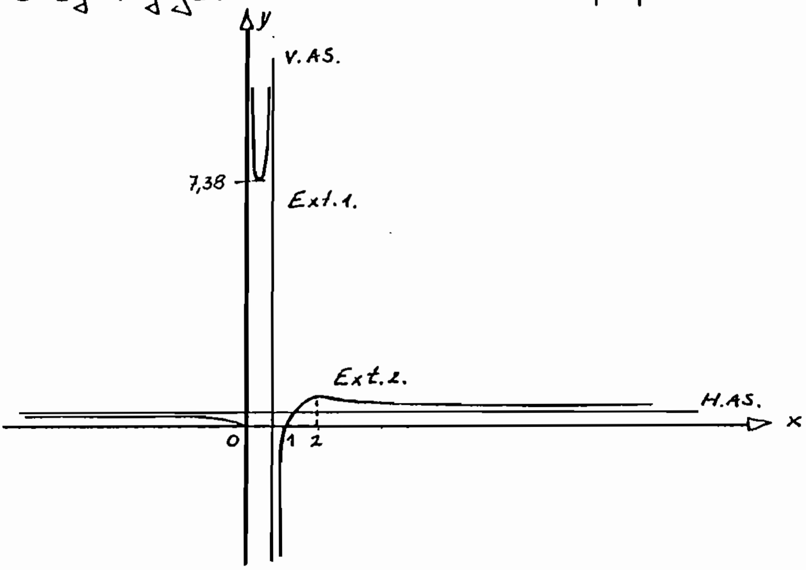
$$\lim_{x\to 0+} \frac{(x-1)}{(3x-2)} e^{\frac{1}{x}} = \frac{1}{2}e^{+\infty} = \underline{+\infty}.$$

$$\lim_{x\to \frac{2}{3}-} \frac{(x-1)e^{1/x}}{(3x-2)} = \frac{-\frac{1}{3}e^{3/2}}{0_-} = \underline{+\infty}.$$

$$\lim_{x\to \frac{2}{3}+} -\frac{\frac{1}{3}e^{3/2}}{0_+} = -\infty$$



8° Grafik f-je:

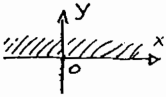


(Z) Nacrtati grafik f-je: $y = \operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x})$.

RJ: $y = \operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x})$.

1° $O(f(x))$: a) $\operatorname{arctg} e^x$: $\forall x \in \mathbb{R}$
b) $\log_a x$: $x>0$
c) $e^{2x} > 0$ $\forall x \in \mathbb{R}$

} $\forall x \in \mathbb{R}$.



2° Ima smisla ispitati simetričnost jer je $O(f)$: simetrično.

$y(-x) = \operatorname{arctg}\frac{1}{e^x} + x + \frac{1}{2}\ln\left(1+\frac{1}{e^{2x}}\right) \neq y(x), \neq -y(x) \Rightarrow$ nije simetrično.

* Uzeti u obzir prilikom traženja asimptota.

3° Nule f-je:

$y=0$ za $\operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x}) = 0$

4° Izvodi:

$$f'(x) = \left(\operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x})\right)' = \left| \begin{array}{l} (\operatorname{arctg} x)' = \frac{1}{1+x^2} \\ (\ln x)' = \frac{1}{x} \end{array} \right| =$$

$$= \frac{1}{1+e^{2x}}(e^x)' - 1 + \frac{1}{2}\frac{1}{(1+e^{2x})}\cdot(1+e^{2x})' =$$

$$= \frac{e^x}{1+e^{2x}} - 1 + \frac{1}{2}\frac{1}{1+e^{2x}}\left((e^x)^2\right)' = \frac{e^x}{1+e^{2x}} - 1 + \frac{1}{2}\cdot\frac{2e^x\cdot e^x}{1+e^{2x}} =$$

$$= \frac{e^x}{1+e^{2x}} + \frac{e^{2x}}{1+e^{2x}} - 1 = \frac{e^{2x}+e^x-1-e^{2x}}{1+e^{2x}} = \frac{e^x-1}{1+e^{2x}}$$

$$f''(x) = \left(\frac{e^x-1}{1+e^{2x}}\right)' = \frac{(e^x-1)'(1+e^{2x}) - (1+e^{2x})'(e^x-1)}{(1+e^{2x})^2} =$$

$$= \frac{e^x(1+e^{2x}) - (2e^x\cdot e^x)(e^x-1)}{(1+e^{2x})^2} = \frac{e^x(1+e^{2x}) - 2e^{2x}(e^x-1)}{(1+e^{2x})^2} =$$

$$= \frac{e^x(1+e^{2x} - 2e^x(e^x-1))}{(1+e^{2x})^2} = \frac{e^x(1+e^{2x} - 2e^{2x} + 2e^x)}{(1+e^{2x})^2} =$$

$$= \frac{e^x(2e^x - e^{2x} + 1)}{(1+e^{2x})^2} = \boxed{\frac{e^x(-e^{2x} + 2e^x + 1)}{(1+e^{2x})^2}}$$

5° Ispitivanje prvog izvoda:

$y'(x) = \frac{e^x - 1}{1+e^{2x}}$

a) $D(f): \forall x \in R$

b) $y'(x) = 0$ za $e^x - 1 = 0 \Rightarrow e^x = 1 \Rightarrow x = 0$

To je S.T.I.

6° Extremi:

$y''(0) = \frac{1(-1+2+1)}{(1+1)^2} = \frac{2}{4} = \frac{1}{2} > 0 \Rightarrow$ minimum.

$y(0) = \text{arctg}\, e^0 - 0 + \frac{1}{2}\ln(1+e^0) = \frac{\pi}{4} + \frac{1}{2}\ln 2 = 1{,}131\ldots$

Ext. 1. (0, 1,131...) minimum.

7° Ispitivanje drugog izvoda:

$y''(x) = \frac{e^x(-e^{2x} + 2e^x + 1)}{(1+e^{2x})^2}$

$y''(x) = 0$ za $(-e^{2x} + 2e^x + 1) = 0 \quad = |e^x = t| \Rightarrow$

$\Rightarrow -t^2 + 2t + 1 = 0, \quad t_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \Rightarrow t_{1,2} = \frac{-2 \pm \sqrt{4+4}}{-2} =$

$= \frac{-2 \pm \sqrt{4 \cdot 2}}{-2} = \frac{-2 \pm 2\sqrt{2}}{-2} \Rightarrow \quad t_1 = 1 - \sqrt{2}, \quad t_2 = 1 + \sqrt{2}$

$e^x = t \Rightarrow x = \log_e t = x = \ln t \Rightarrow \left.\begin{matrix} x_1 = \ln(1-\sqrt{2}) \\ x_2 = \ln(1+\sqrt{2}) \end{matrix}\right\}$

Ovo su S.T. II vrste tj. mogući prevoji.

Provjera: $-e^{2x} + 2e^x + 1 \Rightarrow$

1) $-e^{2\ln(1-\sqrt{2})} + 2e^{(1-\sqrt{2})} + 1 = -e^{\ln(1-\sqrt{2})^2} + 2e^{(1-\sqrt{2})} + 1 =$

$= -(1-\sqrt{2})^2 + 2(1-\sqrt{2}) + 1 = -1 + 2\sqrt{2} - 2 + 2 - 2\sqrt{2} + 1 = 0$ (T)

2) $-(1+\sqrt{2})^2 + 2(1+\sqrt{2}) + 1 = -(1 + 2\sqrt{2} + 2) + 2 + 2\sqrt{2} + 1 = 0$ (T)

$\left.\begin{matrix} x_1 = \ln(1-\sqrt{2}) \\ x_2 = \ln(1+\sqrt{2}) \end{matrix}\right\}$ Pogledamo ova dva izraza:

Prvi izraz ne trebamo provjeravati jer on ne postoji, jer je ln definisano za $x > 0$.

Trebamo provjeriti samo tačku x_2.

9.

$y''(x) = \frac{e^x(-e^{2x}+2e^x+1)}{(1+e^{2x})^2}$, $x_2 = \ln(1+\sqrt{2}) = 0{,}881\ldots$

$y''(0{,}5) = \frac{e^{0,5}(-e+2e^{0,5}+1)}{(1+e)^2} = \frac{2{,}6}{13{,}8} > 0 \Rightarrow$ „U"-konveksna.

$y''(1) = \frac{e(-e^2+2e+1)}{(1+e^2)^2} = \frac{-2{,}589}{70{,}37} < 0 \Rightarrow$ „∩"-konkavna.

Vidimo da drugi izvod u $x = 0{,}881$ mijenja znak pa je ovo prevojna tačka. Koordinate prevojne tačke su:

$y(0{,}881) = \operatorname{arctg} e^{0,881} - 0{,}881 + \frac{1}{2}\ln(1+e^{2\cdot 0,881}) =$

$= 67{,}5^\circ - 0{,}881 + 0{,}96 = 1{,}178 - 0{,}881 + 0{,}96 = 1{,}257.$

Prev. t. $(0{,}881,\ 1{,}257)$.

8° Ispitivanje definicionog područja i asimptota:

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x})\right) =$

$= \lim_{x\to+\infty}\left(\operatorname{arctg} e^x + \frac{1}{2}\ln(1+e^{2x}) - x\right) = \lim_{x\to+\infty}\operatorname{arctg} e^x +$

$+ \frac{1}{2}\lim_{x\to+\infty}\ln(1+e^{2x}) - \lim_{x\to+\infty} x = \frac{\pi}{2} + \underbrace{\infty - \infty}_{\text{neodređeni izraz}} =$

pa moramo to riješiti regularnom transformacijom.

$= \frac{\pi}{2} + \lim_{x\to+\infty}\left(\frac{1}{2}\ln(1+e^{2x}) - x\right) = \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\left(\ln(1+e^{2x}) - 2x\right) =$

$= \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\left(\ln(1+e^{2x}) - \ln e^{2x}\right) = \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\ln\left(\frac{1+e^{2x}}{e^{2x}}\right) =$

$= \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\underbrace{\ln\left(\frac{1}{e^{2x}}+1\right)}_{\to 0} = \frac{\pi}{2}$ ovo je. H.A.S.

ili

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left(\operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x})\right) = \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\ln(1+e^{2x}) -$

$- \lim_{x\to+\infty} x = \frac{\pi}{2} + \overbrace{\infty - \infty}^{\text{ovako nemože}} = \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\left(\ln(1+e^{2x}) - 2x\right) =$

$= \frac{\pi}{2} + \frac{1}{2}\lim\left(\underbrace{\approx 2x - 2x}_{}\right) = \frac{\pi}{2}$ jer: $\ln(1+e^{2x}) \sim \ln e^{2x} \sim 2x$ kad je x dovoljno veliko.

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x})\right) =$$

$$= \lim_{x\to-\infty} \operatorname{arctg} e^x - \lim_{x\to-\infty} x + \frac{1}{2}\lim_{x\to-\infty}\ln(1+e^{2x}) =$$

$$= \lim_{x\to+\infty} \underbrace{\operatorname{arctg}\frac{1}{e^x}}_{\to 0} + \infty + \frac{1}{2}\underbrace{\lim_{x\to+\infty}\ln\left(1+\frac{1}{e^{2x}}\right)}_{\to 0} = +\infty$$

$$* \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\left(\underset{\to 0}{\operatorname{arctg} e^x} - \cancel{x} + \cancel{x}\right) = 0 \quad (\perp)$$

↳ ekvivalenciju ne rodimo jer nakon kraćenja itd., poništavanja ikseva mismo ustvari prihvatili neodređeni izraz $\infty-\infty$, a to ne smijemo uraditi!

Asimptote: k.AS.:

$$a_1 = \lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\underbrace{\frac{\operatorname{arctg} e^x}{x}}_{\to 0} - 1 + \lim_{x\to+\infty}\frac{1}{2}\frac{\ln(1+e^{2x})}{x} =$$

$$= -1 + \frac{1}{2}\lim_{x\to+\infty}\frac{\ln(1+e^{2x})}{x} = \frac{\infty}{\infty} = |\text{Lopital}| =$$

$$= -1 + \frac{1}{2}\lim_{x\to+\infty}\frac{\frac{1}{(1+e^{2x})}\cdot 2e^x\cdot e^x}{1} = -1 + \frac{1}{2}\lim_{x\to+\infty}\frac{2e^{2x}}{1+e^{2x}} =$$

$$= -1 + \lim_{x\to+\infty}\frac{1}{\frac{1}{e^{2x}}+1} = -1 + \frac{1}{2} = \underline{-\frac{1}{2}}.$$

$$a_2 = \lim_{x\to+\infty}\frac{f(x)}{x} = 0 - 1 + \frac{1}{2}\lim_{x\to+\infty}\frac{\ln(1+e^{2x})}{x} = 0 - 1 + \frac{1}{2}\lim_{x\to+\infty}\frac{\sim 2x}{x} =$$

$$= 0 - 1 + \frac{1}{\cancel{2}}\cdot\cancel{2} = 0 - 1 + 1 = 0 \Rightarrow (\perp)$$ ponovo zbog prihvaćanja neodređenog izraza bez regularne transformacije.

$$\lim_{x\to-\infty}(f(x)/x) = \lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{\operatorname{arctg} e^x}{x} - 1 + \lim_{x\to-\infty}\frac{\ln(1+e^{2x})}{2x} =$$

$$= \underbrace{\operatorname{arctg} 0/_{-\infty}}_{\to 0} - 1 + \frac{1}{2}\lim_{x\to-\infty}\frac{\ln(1+e^{2x})}{x} = 0 - 1 + \frac{0}{-\infty} = \underline{-1}.$$

$$b_1 = \lim_{x\to+\infty}(f(x) - a_1 x) = \lim_{x\to+\infty}\left(\operatorname{arctg} e^x + \frac{1}{2}\ln(1+e^{2x}) - x + \frac{1}{2}x\right) =$$

$$= \lim_{x\to+\infty}\underbrace{\operatorname{arctg} e^x}_{\to \pi/2} + \frac{1}{2}\lim_{x\to+\infty}\left(\ln(1+e^{2x}) - \ln e^x\right) =$$

'91.

$$= \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\left(\ln\left(\frac{1+e^{2x}}{e^x}\right)\right) = \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\ln\frac{\frac{1}{e^{2x}}+1}{\frac{1}{e^x}} =$$

$$= \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\ln\frac{1}{\to 0} = \frac{\pi}{2} + \frac{1}{2}\lim_{x\to+\infty}\ln(+\infty) = \underline{+\infty} \Rightarrow$$

Da nemamo kosu asimptotu $y_1 = a_1x + b_1$.

$$b_2 = \lim_{x\to-\infty}(f(x) - a_2x) = \lim_{x\to-\infty}\left(\operatorname{arctg} e^x - x + \frac{1}{2}\ln(1+e^{2x}) + x\right) =$$

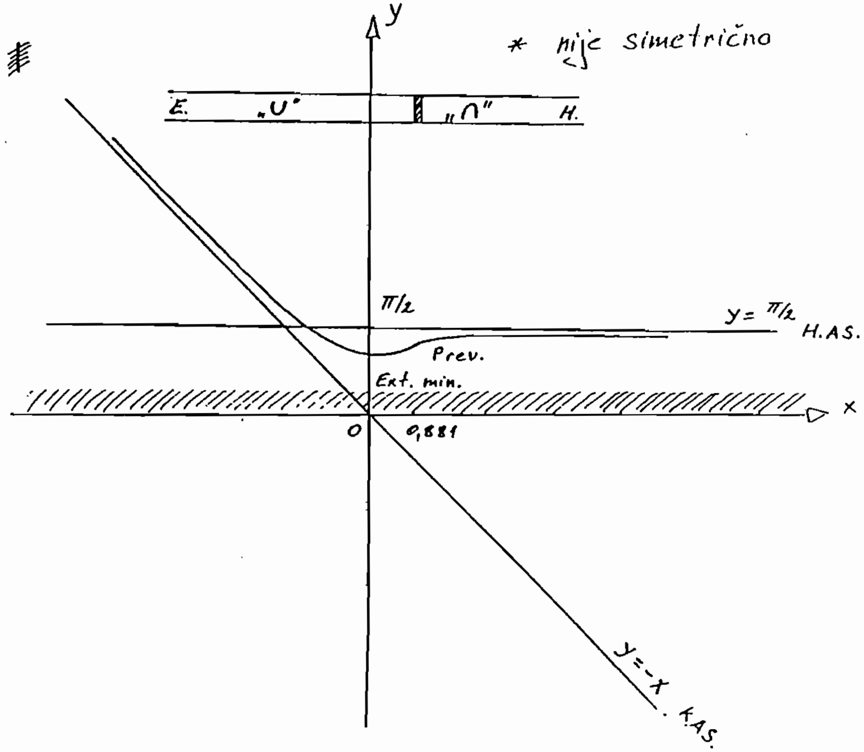
= imamo neodređeni izraz = ali je on istog reda =

$$= \lim_{x\to-\infty}\left(\operatorname{arctg} e^x + \frac{1}{2}\ln(1+e^{2x})\right) = \lim_{x\to+\infty}\left(\underbrace{\operatorname{arctg}\frac{1}{e^x}}_{\to 0}\right) +$$

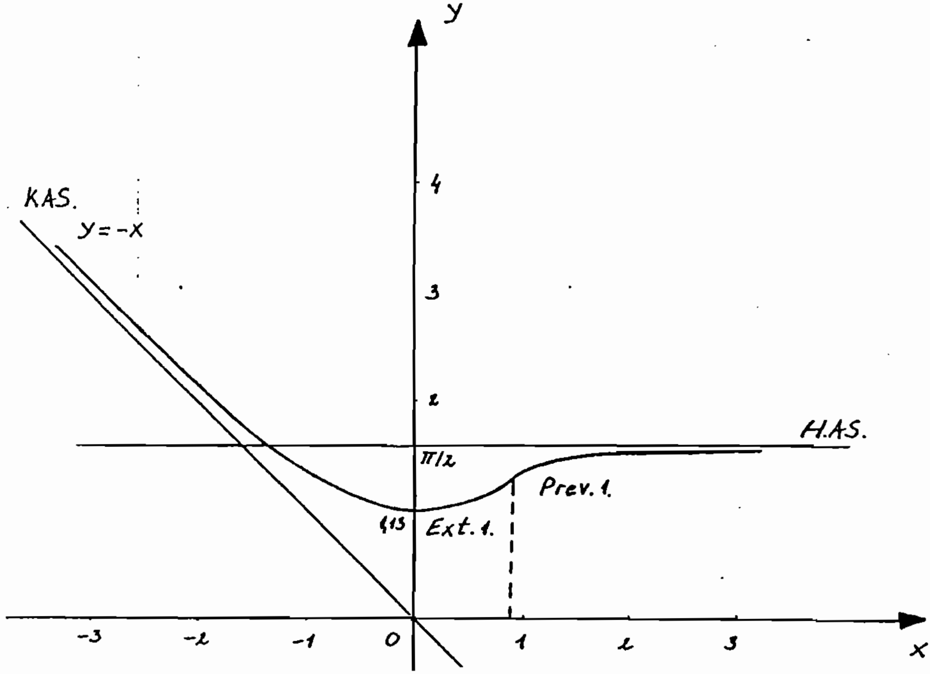
$$+ \frac{1}{2}\lim_{x\to+\infty}\underbrace{\ln\left(1+\frac{1}{e^{2x}}\right)}_{\to 0} = 0$$

$a_2 = -1$, $b_2 = 0 \Rightarrow y_2 = a_2x + b_2 \Rightarrow \boxed{y = -x}$ k.A.S.

9° Napravimo skicu svega što smo našli:



10° Grafik:



(7) Iz familije f-ja zadanih formulom $f_{\alpha,\beta,\gamma}(x) = \alpha \ln x + x^2 + \beta x + \gamma$, $(\alpha,\beta,\gamma \in R)$ izdvojiti onu f-ju koja u točki $x=1$ ima ekstremnu vrijednost jednaku 2, a za $x=3$ ima prevojnu točku. Zatim ispitati tok i nacrtati grafik dobijene f-je.

RJ:

$$f(x) = \alpha \ln x + x^2 + \beta x + \gamma$$

Uslovi za nalaženje koeficijenata α, β i γ su:

* F-ja u $x=1$ ima ekstrem $y=2$.
* F-ja u $x=3$ ima prevojnu točku.

Ovo znači da ako uvrstimo $x=1$, vrijednost f-je mora biti 2, (ovim dobijemo prvi uslov), to nam kaže da moramo imati i bar jednu ST. I. f. da je $y'(x)=0$ za bar jednu točku, (ovim dobijemo drugi

uslov za nalaženje koeficijenata), i iz druge jednakosti vidimo da za bar jedno $x \in D(f)$ mora biti $f''(x)=0$, tj. da imamo bar jednu ST. II.

$f(1)=2 \Rightarrow \alpha \cdot \ln 1 + 1^2 + \beta \cdot 1 + \gamma = 2 \Rightarrow 1+\beta+\gamma=2 \Rightarrow$

$\boxed{\beta+\gamma=1}$ 1)

$f'(x)=0 \Rightarrow \alpha \cdot \frac{1}{x} + 2x + \beta = 0 \Rightarrow \frac{\alpha}{x} + 2x + \beta = 0$

$f'(1)=0 \Rightarrow \alpha + 2 + \beta = 0 \Rightarrow \boxed{\alpha+\beta=-2}$ 2)

$f''(x)=0 \Rightarrow -\frac{\alpha}{x^2} + 2 = 0 \Rightarrow f''(3)=0 \Rightarrow -\frac{\alpha}{9} + 2 = 0 \Rightarrow$

$\Rightarrow \boxed{\alpha=18}$ 3)

Sada možemo naći koeficijente α, β i γ:

$$\left.\begin{array}{l} \alpha+\beta=-2 \Rightarrow \beta=-2-18=-20 \\ \beta+\gamma=1 \Rightarrow \gamma=21 \end{array}\right\} \begin{array}{l} \alpha=18 \\ \beta=-20 \\ \gamma=21 \end{array}$$

Ovim smo iz familije izdvojili sljedeću f-ju:

$y = 18\ln x + x^2 - 20x + 21.$

1° $D(f)$: $\forall x>0$

2° Simetrija: Nema smisla ispitivati simetriju jer je definiciono područje nesimetrično.

3° Nule f-je:

$18\ln x + x^2 - 20x + 21 = 0 \qquad e^{\ln f(x)} = f(x), \ \ln e^{f(x)} = f(x).$

$\ln x^{18} + \ln e^{x^2} - \ln e^{20x} + \ln e^{21} = \ln 1 \Rightarrow$

$\Rightarrow \ln \frac{x^{18} e^{x^2} e^{21}}{e^{20x}} = \ln 1 \Rightarrow \ln(x^{18} \cdot e^{(x^2-20x+21)}) = \ln 1 \Rightarrow$

$\Rightarrow x^{18} \cdot e^{x^2-20x+21} = 1 \Rightarrow$

$\Rightarrow e^{x^2-20x+21} = \frac{1}{x^{18}} \Rightarrow$ da je potrebno nacrtati dvije f-je: $y_1 = e^{x^2-20x+21}$ i $y_2 = \frac{1}{x^{18}}$ i vidjeti gdje se sijeku, u stvari vidjeti da li su presjeci mogući, je ovdje ne možemo egzaktim

putem dobiti rješenja jednačine koja je transcedentna.

$y = \frac{1}{x^{18}}$

$y(-x) = \frac{1}{x^{18}} = y(x)$ – parna je

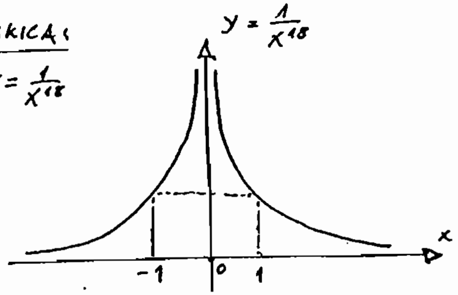
$\lim_{x \to \pm\infty} f(x) = 0$

$\lim_{x \to 0_+} f(x) = +\infty$

$\lim_{x \to 0_-} f(x) = -\infty$

SKICA:

$y = \frac{1}{x^{18}}$



$y = e^{x^2 - 20x + 2}$ zavisi od $x^2 - 20x + 2$

$x^2 - 2x + 2 = 0 \Rightarrow x_{1,2} = \frac{20 \pm 19{,}8}{2}$

$x_1 = 0{,}1 \Rightarrow y = 1$

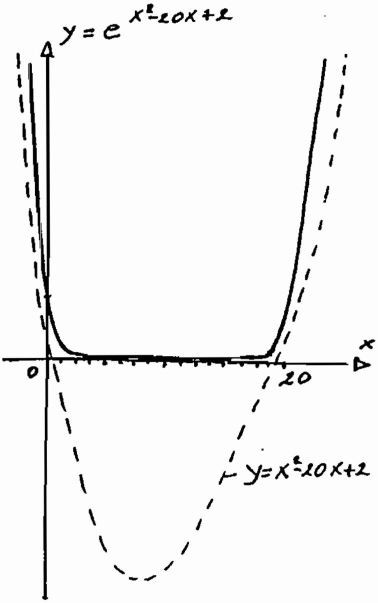
$x_2 = 19{,}9 \Rightarrow y = 1$

$\lim_{x \to -\infty} e^{x^2 - 20x + 2} = e^{+\infty} = +\infty$

$\lim_{x \to +\infty} e^{x^2 - 20x + 2} = +\infty$

$\lim_{x \to 0_+} e^{x^2 - 20x + 2} = e^2 \simeq 7{,}4$

$\lim_{x \to 0_-} e^{x^2 - 20x + 2} \simeq 7{,}4.$



Vidimo da možemo imati bar dva presjeka ove dvije krive.

Znači, nula ima ali ih egzaktnim putem nismo mogli dobiti.

4° Izvodi: $y = 18 \ln x + x^2 - 20x + 21.$

$y' = \frac{18}{x} + 2x - 20$

$y'' = -\frac{18}{x^2} + 2$

Ispitajmo sada izvode:

$f'(x) = \frac{18}{x} + 2x - 20 \Rightarrow x \neq 0$ – stac. tačka je $x=0$ ali pošto ne ulazi u def. područje f-je otpada.

$\frac{18}{x}+2x-20=0 \,/\cdot x,\; x\neq 0$

$18+2x^2-20x=0 \Rightarrow 2x^2-20x+18=0 \,/:2 \Rightarrow \underline{x^2-10x+9=0}$

$x_{1,2}=\frac{10\pm\sqrt{100-36}}{2} \Rightarrow x_{1,2}=\frac{10\pm 8}{2},\; x_1=9,\; x_2=1.$

$\left.\begin{matrix} x_1=9 \\ x_2=1 \end{matrix}\right\}$ ovo su ST.I. Za x_2 znamo da je ekstrem. Potrebno je provjeriti samo tačku $x_1=9$.

$y''_{(x)}=-\frac{18}{x^2}+2$

$y''(9)=-\frac{18}{81}+2=-0{,}2+2>0 \Rightarrow$ minimum

$y(9)=18\cdot\ln 9+81-20\cdot 9+21=39{,}6+81-180+21=\underline{-38{,}4}$

Imamo dva ekstrema:

$\underline{Ext.\,1.\,(1,2)\ \max.}$

$\underline{Ext.\,2.\,(9,-38{,}4)\ \min.}$

$y''_{(x)}=-\frac{18}{x^2}+2 \Rightarrow -\frac{18}{x^2}+2=0 \,/\cdot x^2,\; x\neq 0 \Rightarrow$

$\Rightarrow -18+2x^2=0 \Rightarrow 2x^2=18 \Rightarrow x_{1,2}=\pm 3.\} \Rightarrow$ ST.II.

tačka $x=-3$ ne može biti kandidat za prev. tačku jer ne ulazi u def. područje f-je.

$x=3 \Rightarrow y(3)=18\ln 3+9-60+21=-10{,}2$

$\underline{Prev.\,1.\,(3,-10{,}2)}$

5° Ispitajmo rast i opadanje f-je:

$y'_{(x)}=\frac{18}{x}+2x-20=0 \Rightarrow \frac{18}{x}+2x-20=0 \,/\cdot x \Rightarrow$

$\Rightarrow 18+2x^2-20x=0 \Rightarrow 2x^2-20x+18=0 \Rightarrow x^2-10x+9=0$

$x_{1,2}=9,3.$ TABLICA:

$\Rightarrow$

x−3	− 0	+	+
x−9	−	− 0	+
	(+)	(−)	(+)
	↗	↘	↗

6° Ispitajmo konveksnost i konkavnost:

$$y'' = -\frac{18}{x^2} + 2 = 0 \qquad x = \pm 3$$

TABLICA:

	0	3	$+\infty$
$x-3$	$-$	0	$+$
	„$\cap$"		„$\cup$"
	konkavna		konveksna.

7° Ispitajmo definiciono područje:

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}(18\ln x + x^2 - 20x + 21) = \underline{+\infty}.$$

$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_+}(18\ln x + x^2 - 20x + 21) = \underline{-\infty}.$$

Ovo je vertikalna asimptota: $\underline{x=0}$.

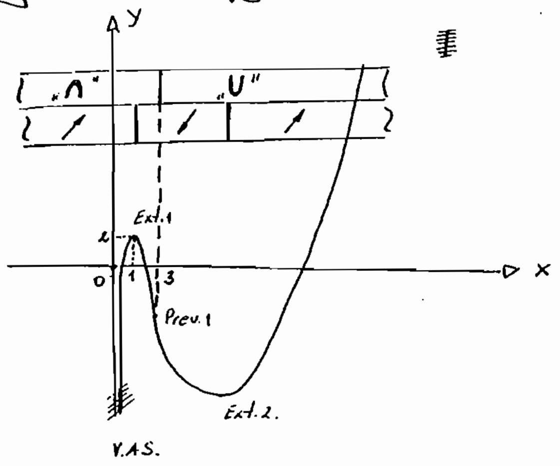
8° Ispitajmo asimptote

$$\lim_{x\to+\infty}\frac{f(x)}{x} = \lim_{x\to+\infty}\frac{18\ln x + x^2 - 20x + 21}{x} = \frac{\infty}{\infty} = |\text{Z.R.T}| = |\text{Lopital}| =$$

$$= \lim_{x\to+\infty}\frac{\frac{18}{x} + 2x - 20}{1} = \underline{+\infty}. \quad \underline{\text{Nema k.A.S.}}$$

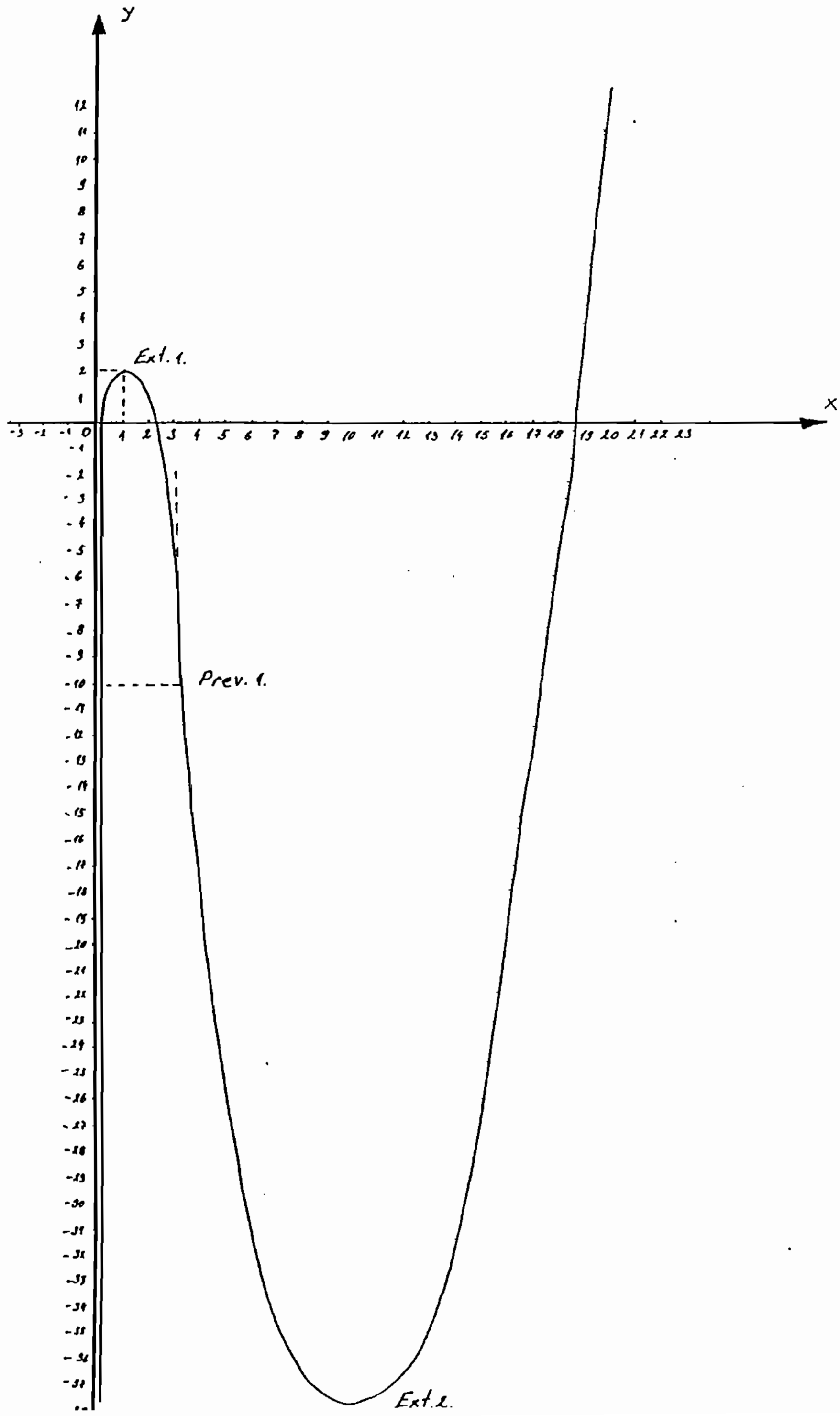
Nismo ispitivali limese kada $x\to-\infty$ jer je f-ja definisana za $x>0$, pa to nema smisla raditi.

9° Nacrtajmo skicu f-je:



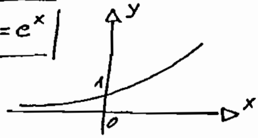
Integrali

10. Grafik f-je:

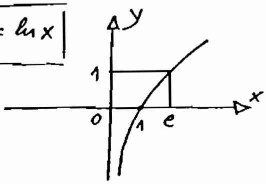


OSNOVNE F-je:

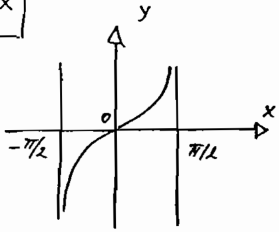
$y = e^x$



$y = \ln x$

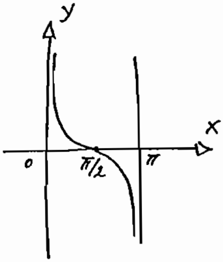


$y = tg\, x$



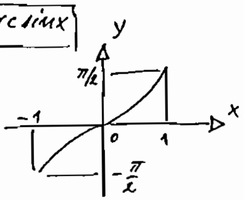
$D(y): \forall x \in R \setminus \{(2k+1)\frac{\pi}{2}\}, k \in Z$

$y = ctg\, x$



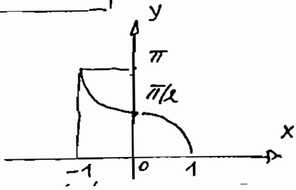
$D(y): \forall x \in R \setminus \{k\pi\}, k \in Z$

$y = \arcsin x$

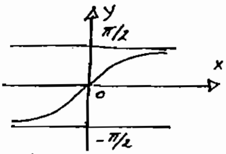


neprekidna na $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \arccos x$

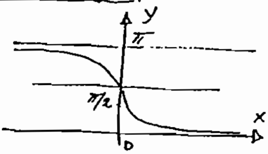


$y = arctg\, x$

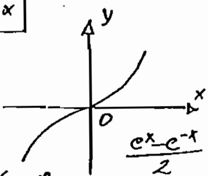


neprekidna $\forall x \in R$

$y = arcctg\, x$



$y = sh\, x$

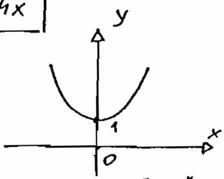


$\frac{e^x - e^{-x}}{2}$

$D(y): \forall x \in R$

$R(y): (+\infty, -\infty)$.

$y = ch\, x$

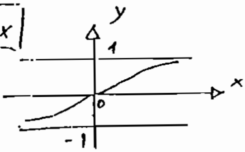


$D(y): \forall x \in R$

$\frac{e^x + e^{-x}}{2}$

$R(y): [1, +\infty)$.

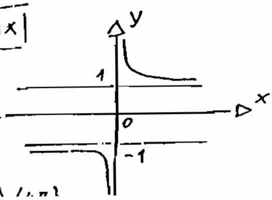
$y = th\, x$



$D(f): \forall x \in R$

$R(f): [-1, 1]$.

$y = cth\, x$



$\forall x \in R \setminus \{...\}$

$ch^2 x = 1 + sh^2 x$

$sh^2 x = ch^2 x - 1$

INTEGRALI:

$$\int dx = x + C$$

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin(nx) dx = -\frac{\cos(nx)}{n} + C$$

$$\int \cos(nx) dx = \frac{\sin(nx)}{n} + C$$

$$\int \frac{dx}{\cos^2 x} = \operatorname{tg} x + C$$

$$\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C$$

$$\int \operatorname{sh} x \, dx = \operatorname{ch} x + C$$

$$\int \operatorname{ch} x \, dx = \operatorname{sh} x + C$$

$$\int \frac{dx}{\operatorname{ch}^2 x} = \operatorname{th} x + C$$

$$\int \frac{dx}{\operatorname{sh}^2 x} = -\operatorname{cth} x + C$$

$$\int e^x dx = e^x + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C$$

$$\int \frac{dx}{x \pm a} = \ln|x \pm a| + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$$

$$\int \frac{dx}{\sqrt{k^2 - x^2}} = \arcsin \frac{x}{k} + C$$

$$\int \frac{dx}{x^2 - k^2} = \frac{1}{2k} \ln \frac{k-x}{k+x} + C, \quad |x| < |k|$$

$$= \frac{1}{2k} \ln \frac{x-k}{x+k} + C, \quad |x| > |k|$$

$$\int \frac{dx}{k^2 + x^2} = \frac{1}{k} \operatorname{arctg} \frac{x}{k} + C$$

$$\int \frac{dx}{\sqrt{k^2 + K^2}} = \ln(x + \sqrt{x^2 + k^2}) + C_1$$

$$\int \frac{dx}{\pm\sqrt{x^2 - 1}} = \ln(x \pm \sqrt{x^2 - 1}) + C$$

$$\int \frac{dx}{\pm\sqrt{x^2 - k^2}} = \ln(x \pm \sqrt{x^2 - k^2}) + C$$

$$\int \frac{dx}{k^2 - x^2} = \begin{cases} \frac{1}{2k} \ln \frac{k+x}{k-x}, & |x| < |k| \\ \frac{1}{2k} \ln \frac{x+k}{x-k}, & |x| > |k| \end{cases}$$

$$\int \frac{dx}{\sin x} = \ln\left|\operatorname{tg} \frac{x}{2}\right| + C$$

$$\int \frac{dx}{\cos x} = \ln\left|\operatorname{tg}\left(\frac{x}{2} + \frac{\pi}{4}\right)\right| + C$$

$$\int \frac{dx}{bx \pm a} = \frac{1}{b} \ln|bx \pm a| + C$$

IZVODI:

$$x^n = nx^{n-1}$$

$$\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

$$\sin x = \cos x$$

$$\cos x = -\sin x$$

$$\operatorname{tg} x = \frac{1}{\cos^2 x}$$

$$\operatorname{ctg} x = -\frac{1}{\sin^2 x}$$

$$\arcsin x = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$

$$\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\operatorname{arctg} x = \frac{1}{1+x^2}$$

$$\operatorname{arcctg} x = -\frac{1}{1+x^2}$$

$$a^x = a^x \ln a, \quad a > 0$$

$$e^x = e^x$$

$$\ln x = \frac{1}{x} \qquad \left(\log_a x\right)' = \frac{1}{x} \cdot \ln a$$

$$\operatorname{sh} x = \operatorname{ch} x$$

$$\operatorname{ch} x = \operatorname{sh} x$$

$$\operatorname{th} x = \frac{1}{\operatorname{ch}^2 x}$$

$$\operatorname{cth} x = -\frac{1}{\operatorname{sh}^2 x}$$

$$\operatorname{Arsh} x = \frac{1}{\sqrt{1+x^2}}$$

$$\operatorname{Arch} x = \frac{1}{\sqrt{x^2-1}}, \quad |x| > 1$$

$$\operatorname{Arth} x = \frac{1}{1-x^2}, \quad |x| < 1$$

$$\operatorname{Arcth} x = -\frac{1}{x^2-1}, \quad |x| > 1$$

Predtip A:

$\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{\text{kvadratna f-ja}}$ štimamo na: $\int \frac{dx}{k^2+x^2} = \frac{1}{k} \operatorname{arctg} \frac{x}{k} + c$

$\int \frac{dx}{x^2-k^2} = \frac{1}{2k} \ln \frac{k-x}{k+x} + c.$

* Radimo nadopunu po formuli:

$$x^2 \pm px \pm q = \left(x \pm \frac{p}{2}\right)^2 - \left(\frac{p}{2}\right)^2 \pm q.$$

Predtip B:

$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \int \frac{dx}{\sqrt{\text{kvadratna f-ja}}} \Rightarrow \int \frac{dx}{\sqrt{k^2-x^2}} = \arcsin \frac{x}{k} + C$

$$\int \frac{dx}{\sqrt{k^2+x^2}} = \ln\left(x + \sqrt{k^2+x^2}\right) + C$$

$$\int \frac{dx}{\sqrt{x^2-k^2}} = \ln\left(x + \sqrt{x^2-k^2}\right) + C$$

Predtip C:

$\int \sqrt{ax^2+bx+C} \cdot dx = \int \sqrt{\text{kvad. f-ja}} \cdot dx \Rightarrow \int \sqrt{t^2-k^2}\, dt$, $\int \sqrt{k^2-t^2}\, dt$, $\int \sqrt{t^2+k^2}\, dt$

množimo sa istim i radimo parcijalnu integraciju

Tip I:

$$\int R(x) \cdot dx = \int \frac{P_n(x)}{Q_m(x)} = \int (\text{razlomljene rac. f-je od } x) \cdot dx$$

* x nigdje nije pod korjenom.

Postupak:

1) izlučiti koeficijent najviše potencije od x u nazivniku
2) dovesti razlomljenu f-ju na pravu racionalnu razlomljenu f-ju:
   - $P_n(x)$ ima isti stepen kao $Q_m(x)$: tada dijelimo: $\frac{P_n}{Q_m}$!
   - $P_n(x)$ ima manji stepen od $Q_m(x)$: tada je već prava rac. razlomljena f-ja i ne treba ništa dijeliti!
   - $P_n(x)$ ima veći stepen od $Q_m(x)$: tada dijelimo: $\frac{P_n}{Q_m}$!
3) rastavljamo izraz na parcijalne razlomke, tj. na faktore oblika $(x \pm a)$, a to se vrši određivanjem nultačaka nazivnika.

* svaka realna nultočka nazivnika daje jedan parcijalni razlomak oblika: $\frac{\text{konstanta}}{\text{linearni faktor}} = \frac{A}{x \pm a}$

3a) nultočke nazivnika su realne, a nisu sve različite, tj., nazivnik ima višestruke realne nule:

Tada linearni faktor koji potiče od n-tastruke linearne tačke nazivnika $(x-a)^n$ daje n parcijalnih razlomaka u kojim su brojnici konstante A, B, C, ... a nazivnici faktori: $(x-a)^n, (x-a)^{n-1}, \ldots$

3b) nazivnik integranta ima također kompleksne nule koje uvijek dolaze u paru kao konjugirano kompleksne: Svaki par konjugirano kompleksnih nultačaka nazivnika daje jedan parcijalni razlomak oblika:

$$\frac{Ax+B}{\text{kvadratna f-ja, kojoj su nultačke konjugovano kompleksne}}$$

Kada nazivnik ima kompleksne nultočke broj parcijalnih razlomaka je manji od broja nultačaka odnosno stepena polinoma u nazivniku, ali je broj konstanata A, B, C, uvijek jednak stepenu polinoma.

3c) nazivnik ima realne i kompleksne tačke, pri čemu su svi ili neki parovi konjugirano kompleksnih nultačaka višestruki:

Isto kao kod višestrukih realnih tačaka, broj konstanti jednak je stepenu polinoma.

Tip II:

$\int R(x, \sqrt[n]{ax+b})\,dx$; $\int R\left(x, \sqrt[n]{\frac{ax+b}{cx+d}}\right)dx$; R - neka razlomljena racionalna f-ja.

U ovoj tip spadaju svi izrazi čiji su integranti sastavljeni od: konstanata, x, $\sqrt[n]{ax+b}$, $\sqrt[n]{\frac{ax+b}{cx+d}}$ između kojih su racionalne operacije: zbrajanje, oduzimanje, množenje i dijeljenje.

* pod korjenom je linearna f-je, odnosno koeficijent linearnih f-ja!

Postupak:

1) Supstitucija je: $\sqrt[n]{ax+b}=t$, odnosno $\sqrt[n]{\frac{ax+b}{cx+d}}=t$

2) Izvlačimo x i dx i uvrštavamo ga da bi dobili tip I.

3) kada dobijemo rezultat vratimo smjenu na x.

1a) Ima slučaj kada imamo više korjena u integrantu npr: $\frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x}}$. U tom slučaju smjena je: $t^n=x$; gdje je „n" najmanji zajednički sadržalac nazivnika stepena nad x. konkretno kod nas to je broj 12, jer u nazivnicima imamo 2, 3 i 4 iz $x^{1/2}$, $x^{1/3}$ i $x^{1/4}$.
Nakon ovoga se dobije racionalna f-ja koja je ustvari Tip I.

Tip III

$\int R(x,\sqrt{ax^2+bx+c})dx$ – integrant je racionalna f-ja od x i drugog korjena iz kvadratne f-je sa konačnim brojem racionalnih operacija.

* u integrantu je uvijek drugi korjen iz kvadratne f-je.

a) „a" u $\sqrt{ax^2+bx+c}$ je pozitivan br., $a>0$.

Postupak:

1) supstitucija : $\sqrt{ax^2+bx+c}=\sqrt{a}\cdot x+t$

2) izvučemo x odavde i onda ga uvrstimo u supstituciju pa dobijemo $\sqrt{ax^2+bx+c}$ = nešto

3) izračunamo dx i sve uvrstimo pa dobijemo Tip I.

b) „a" u $\sqrt{ax^2+bx+c}$ je negativan br., $a<0$

Postupak:

1) Odredimo nultačke (x_1, x_2) kvadratne f-je pod korjenom: $ax^2+bx+c=a(x-x_1)(x-x_2)$

2) supstitucija : $a=-a$ u smislu: $\sqrt{-a(x-x_1)(x-x_2)}=\sqrt{a(x_1-x)(x-x_2)}=$
$=t(x_1-x)$ ili $t(x-x_2)$

3) sve svodimo na t isto u prethodnom slučaju.

c) U brojniku je cijela racionalna f-ja, a u nazivniku II korjen kvadratne f-je.

$$I=\int\frac{a_nx^n+a_{n-1}x^{n-1}+\ldots+a_3x^3+a_2x^2+a_1x+a_0}{\sqrt{ax^2+bx+c}}dx$$

1) Stavimo ovo:

$$I=(A_{n-1}x^{n-1}+A_{n-2}x^{n-2}+\ldots+A_1x+A_0)\sqrt{ax^2+bx+c}+A_n\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

$A_n, A_{n-1}, \ldots, A_1, A_0$ – nepoznate konstante.

2) Tražimo izvod obe strane po x pamteći da je

$\left(\int\frac{dx}{\sqrt{ax^2+bx+c}}\right)'=\frac{1}{\sqrt{ax^2+bx+c}}$ – Tako dobijemo dva identična polinoma. Poredimo ih i dobijemo n+1 lin. jednačinu, čime dobijemo nepoznate konstante.

d) U brojniku imamo:

$$\int(a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0)\sqrt{ax^2+bx+c}\,dx$$

Postupak:

1) Množenjem i dijeljenjem sa $\sqrt{ax^2+bx+c}$ svodimo ga na prethodni oblik.

e) Imamo oblik:

$$\int\frac{dx}{(x-k)^n\sqrt{ax^2+bx+c}}\quad, n\in\mathbb{N}, k\in\mathbb{R}\ (\text{često nula})$$

1) supstitucija : $x-k=\frac{1}{t}$ – time ga svodimo na osnovni integral.

f) Imamo nelinearan faktor: (x^n-k) u nazivniku ispred $\sqrt{ax^2+bx+c}$.

Postupak:

1) Posebno rastavljamo (x^n-k) u parcijalne razlomke i integral rastavljamo u više integrala.

2) Na isti način se računa integral kod kojeg je podintegralna f-ja : $\frac{1}{\sqrt{ax^2+bx+c}}$ pomnožena sa nekom razlomljenom racionalnom f-jom. $R(x)=\frac{P_n(x)}{Q_m(x)}$ – R(x) - rastavljamo na parcijalne razlomke.

Tip IV. (Binomni integrali)

$\int x^p(ax^q+b)^r dx$, $a,b \in R$; $p,q,r \in Q$

Postupak:

Integral je rješiv u dva slučaja:

1) Ako je $\frac{p+1}{q}$ cio broj ili nula $\Rightarrow$ smjena: $ax^q+b=t^s$

s - je nazivnik u eksponentu korjena, tj., u r, ali sa pozitivnim predznakom.

2) Ako je $\frac{p+1}{q}+r$ cio broj ili nula $\Rightarrow ax^q+b=t^s x^q$

Tip V. (Eliptički integrali)

Nerješivi su elementarnim postupkom. Integrant se ne može racionalizirati jer pod korjenom imamo polinom III i višeg stepena. Ovi integrali rješavaju se tako što se integrat rastavi na beskonačno konvergentan red, pa se integrira član po član. Također, svaki eliptički integral može se napisati kao kombinacija tri osnovna eliptička (Langraževa integrala).

Oblik im je:

$$I=\int R\left(x,\sqrt{a_4x^4+a_3x^3+a_2x^2+a_1x+a_0}\right)dx.$$

Integrali transcedentnih funkcija

Tip VI.

$I=\int R(\sin x,\cos x)\,dx$

* u ovaj tip spadaju integrali u koji ulaze sinx, cosx, konstante s tim da su integranti racionalni i sam x ne ulazi u integrant. Međutim ako x uđe u integrant treba pokušati parcijalnom integracijom.

1) Supstitucije su: $tg\frac{x}{2}=t \Rightarrow \sin x=\frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{2t}{1+t^2}$

$\Rightarrow \cos x = \dfrac{1-t^2}{1+t^2}$, $dx = \dfrac{2dt}{1+t^2}$

2) Ulaze li u integrant potencije $\sin x$ i $\cos x$ primjena supstitucije $tg\frac{x}{2}=t$ vodi do složenih integrala tipa I.
Tada postupamo ovako:

2a) Ako je podintegralna f-ja parna u $\sin x$ i $\cos x$ tj.
$R(-\sin x, -\cos x) = R(\sin x, \cos x)$ supstitucija je: $tg x = t$, $x = arctg t$.

2b) Ako je podintegralna f-ja neparna u $\sin x$, tj.
$R(-\sin x, \cos x) = -R(\sin x, \cos x)$ supstitucija je: $\cos x = t$.

2c) Ako je integrant neparna f-ja u $\cos x$ tj.
$R(\sin x, -\cos x) = -R(\sin x, \cos x)$ supstitucija je: $\sin x = t$.

* $\dfrac{1}{\cos^2 x} = 1 + tg^2 x$

$I = \int R(shx, chx)\,dx$

1) supstitucija je: $th\frac{x}{2} = t \Rightarrow \left.\begin{array}{l} shx = \dfrac{2t}{1-t^2} \\ chx = \dfrac{1+t^2}{1-t^2} \end{array}\right\} dx = \dfrac{2dt}{1-t^2}$

2) $\left.\begin{array}{l} chx = \dfrac{e^x+e^{-x}}{2} \\ shx = \dfrac{e^x-e^{-x}}{2} \end{array}\right\} e^x = t,\ dt = e^x dx = t\,dx$

* Potrebne formule:

$chx = ch^2\frac{x}{2} + sh^2\frac{x}{2} = (1 + th^2\frac{x}{2}) \cdot ch^2\frac{x}{2} = \dfrac{1+th^2\frac{x}{2}}{1-th^2\frac{x}{2}}$

$sh x = 2sh\frac{x}{2} \cdot ch\frac{x}{2} = 2th\frac{x}{2} \cdot ch^2\frac{x}{2} = 2th\frac{x}{2} \cdot \dfrac{1}{1-th^2\frac{x}{2}}$

$\sin x = 2\sin\frac{x}{2} \cdot \cos\frac{x}{2} = 2tg\frac{x}{2} \cdot \cos^2\frac{x}{2} = \dfrac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = (1 - tg^2\frac{x}{2}) \cdot \cos^2\frac{x}{2} = \dfrac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}$

Tip VII

$\left.\begin{array}{l} I = \int \sin(mx)\cos(nx)\,dx \\ I = \int \cos(mx)\cos(nx)\,dx \\ I = \int \sin(mx)\sin(nx)\,dx \end{array}\right\}$

$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$

$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$

$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

Potrebno je znati i formule:

$\sin(\alpha \pm \beta) = \sin\alpha \cdot \cos\beta \pm \cos\alpha \cdot \sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \cdot \sin\beta$

Tip VIII.

$I_n = \int \sin^n x\,dx$, $I_n = \int \cos^n x\,dx$, n - cio pozitivan broj.

Postupak:

Integrant rastavljamo u faktore, tako da je drugi faktor $\sin x\,dx$, odnosno $\cos x\,dx$. Iza toga parcijalno integriramo pri čemu računajući prvi integral uzimamo $\cos^2 x = 1-\sin^2 x$, a za drugi integral $\sin^2 x = 1-\cos^2 x$. Oznaci integrala „I" dodaje se indeks koji označava stepen n.

Formule rekurzije koje se dobiju su:

$$\left.\begin{aligned} I_n &= \int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cdot\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx \\ I_n &= \int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\cdot\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx \end{aligned}\right\} n \in N.$$

Ako imamo neparan eksponent podintegralne f-je, postupak rješavanja je skoro isti kao gore navedeni samo ubacimo smjenu $\cos x = t$ ili $\sin x = t$ i ove integrale je lakše rješiti.

Tip IX.

$$\left.\begin{aligned} I &= \int tg^n x\,dx \Rightarrow tg x = t,\ x = arctg\,t,\ dx = \frac{dt}{1+t^2} \\ I &= \int ctg^n x\,dx \Rightarrow ctg x = t,\ x = arcctg\,t,\ dx = -\frac{dt}{1+t^2} \end{aligned}\right\} n \in N$$

$*\quad 1+tg^2 x = \frac{1}{\cos^2 x}$, $\ln 1 = 0$, $arcctg(ctg x) = x$.

Tip X.

$I_n = \int \frac{dx}{\sin^n x}$; $I_n = \int \frac{dx}{\cos^n x}$ - n je cio pozitivan broj.

Postupak:

Integrant rastavljamo u dva faktora tako da drugi faktor bude $\frac{dx}{\sin^2 x}$ odnosno $\frac{dx}{\cos^2 x}$. Onda ide parcijalna integracija pri čemu računajući prvi integral uzimamo:

$ctg x = \frac{\cos x}{\sin x}$ i $\cos^2 x = 1 - \sin^2 x$, a za drugi integral

$tg x = \frac{\sin x}{\cos x}$; $\sin^2 x = 1 - \cos^2 x$.

Dobijemo sljedeće rekurzivne formule:

$$I_n = \int \frac{dx}{\sin^n x} = -\frac{1}{n-1}\frac{\cos x}{\sin^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\sin^{n-2} x}$$

$$I_n = \int \frac{dx}{\cos^n x} = \frac{1}{n-1}\frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1}\int \frac{dx}{\cos^{n-2} x}$$

Tip XI.

$\int \sin^m x \cdot \cos^n x \, dx$, m i n - cijeli eksponenti (pozitivni ili neg.)

1) Oba eksponenta su pozitivna, jedan je paran a drugi neparan:
   Štimamo da od onoga koji je neparan napravimo proizvod ostatka sa $\cos x\, dx$ ili $\sin x\, dx$ pa uzimamo smjenu $\sin x = t$ ili $\cos x = t$.

2) m ili n je pozitivan i neparan, a drugi je negativan:
   Tada vrstu brojnika štimamo u vrstu nazivnika, s tim da u brojniku ostavljamo $\sin x dx$ ili $\cos x dx$, onda idu smjene: $\cos x = t$ ili $\sin x = t$

3) m ili n je pozitivan i paran, a drugi je negativan.
   Sve štimamo da je kompatibilno nazivniku i imamo tip X.

4) m = n, pozitivni i neparni:
   Smjena: $\sin x = t$, i štimamo da je $\cos x\, dx$ u brojniku.

5) m = n, pozitivni i parni:
   $2\sin x \cos x = \sin 2x$, $2x = t \Rightarrow$ tip VIII.

6) m = n, negativni, parni ili neparni:
   $2\sin x \cos x = \sin 2x$, $2x = t \Rightarrow$ tip X.

7) $m \neq n$, pozitivni i parni.
   Podintegralnu veličinu se razdvaja na dio koji ide sa dx tj., na $\sin^2 x dx$ ili $\cos^2 x dx$ i slično i dio koji se transformiše formulom: $2\sin^2 x = \sin 2x$, a prvi dio se transformiše formulom: $2\sin^2 x = 1 - \cos 2x$. Zatim ide

$2x=t$ i $\sin 2x=z$.

Integrali ovog tipa mogu se riješiti i preko integrala tipa VIII. Naštimamo na istu promjenjivu i dobijemo tip VIII.

8) $m \neq n$, negativni i parni:

$$\frac{1}{\sin^2 x} = 1 + \operatorname{ctg}^2 x \Rightarrow \frac{1}{\sin^2 x} = \frac{1+\operatorname{tg}^2 x}{\operatorname{tg}^2 x}$$

drugi dio sređivamo na oblik $\frac{dx}{\cos^2 x}$

$$\frac{1}{\cos^2 x} = 1 + \operatorname{tg}^2 x \Rightarrow \frac{dx}{\cos^2 x} = dt$$

$\operatorname{tg} x = t$

$\frac{dx}{\cos^2 x} = dt$

9) $m \neq n$ negativni i neparni.

Brojnik i nazivnik množimo sa $\cos x \Rightarrow$ smjena $\sin x = t$, $\cos x\,dx = dt$.

10) m ili n je paran, a drugi neparan, ali su oba negativna.

Isto kao pod 9). Množimo sa $\sin x$, $\cos x = t$, itd.

(Z) $$\int\frac{dx}{4x^2+4x+5}=\frac{1}{4}\int\frac{dx}{x^2+x+\frac{5}{4}}=\left|x^2\pm px+q=\left(x\pm\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2+q\right|=$$

$$=\frac{1}{4}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2-\frac{1}{4}+\frac{5}{4}}=\frac{1}{4}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+1}=\left|x+\frac{1}{2}=t,\ dx=dt\right|=$$

$$=\frac{1}{4}\int\frac{dt}{t^2+1}=\frac{1}{4}\operatorname{arctg}t=\frac{1}{4}\operatorname{arctg}\left(x+\frac{1}{2}\right)=\frac{1}{4}\operatorname{arctg}\frac{2x+1}{2}+C.$$

(Z) $$\int\frac{dx}{\sqrt{1+3x+7x^2}}=\frac{1}{\sqrt{7}}\int\frac{dx}{\sqrt{-\left(x^2-\frac{3}{7}x-\frac{1}{7}\right)}}=\frac{1}{\sqrt{7}}\int\frac{dx}{\sqrt{-\left[\left(x-\frac{3}{14}\right)^2-\frac{9}{196}-\frac{1}{7}\right]}}=$$

$$=\frac{1}{\sqrt{7}}\int\frac{dx}{\sqrt{-\left[\left(x-\frac{3}{14}\right)^2-\frac{37}{196}\right]}}=\frac{1}{\sqrt{7}}\int\frac{dx}{\sqrt{\frac{37}{196}-\left(x-\frac{3}{14}\right)^2}}=\left|\frac{37}{196}=k^2,\ k=\frac{\sqrt{37}}{14}\right.$$

$$\left|\begin{matrix}x-\frac{3}{14}=t\\ dx=dt\end{matrix}\right|=\frac{1}{\sqrt{7}}\int\frac{dx}{\sqrt{k^2-t^2}}=\frac{1}{\sqrt{7}}\arcsin\frac{t}{k}=\frac{1}{\sqrt{7}}\arcsin\frac{x-\frac{3}{14}}{\frac{\sqrt{37}}{14}}=$$

$$=\frac{1}{\sqrt{7}}\arcsin\frac{14x-3}{\sqrt{37}}+C.$$

(Z) $$I=\int\sqrt{3x^2+10x+9}\cdot dx=\sqrt{3}\int\sqrt{x^2+\frac{10}{3}x+3}\,dx=\sqrt{3}\int\sqrt{\left(x+\frac{5}{3}\right)^2-\frac{25}{9}+3}\,dx=$$

$$=\sqrt{3}\int\sqrt{\left(x+\frac{5}{3}\right)^2+\frac{2}{9}}\cdot dx=\left|\begin{matrix}x+\frac{5}{3}=t,\ dx=dt\\ \frac{2}{9}=k^2,\ k=\frac{\sqrt{2}}{3}\end{matrix}\right|\Rightarrow$$

$$I=\sqrt{3}\underbrace{\int\sqrt{t^2+k^2}\cdot dt}$$

$$I_1=\int\sqrt{t^2+k^2}\,dt\quad\Rightarrow\quad I=\sqrt{3}\,I_1$$

$$I_1=\int\frac{t^2+k^2}{\sqrt{t^2+k^2}}\,dt=\underbrace{\int\frac{t^2dt}{\sqrt{t^2+k^2}}}_{I_2}+k^2\underbrace{\int\frac{dt}{\sqrt{t^2+k^2}}}_{I_3}$$

$$I_1=I_2+k^2\cdot I_3.$$

I_3 - je tablični neodređeni integral.

$$I_2 = \int t \cdot \frac{t\,dt}{\sqrt{t^2+k^2}} = \left| u = t,\ du = dt;\ dv = \frac{t\,dt}{\sqrt{t^2+k^2}} \right| =$$

$$v = \int \frac{t\,dt}{\sqrt{t^2+k^2}} = \left(t^2+k^2 = z,\ 2t\,dt = dz,\ t\,dt = \tfrac{1}{2}dz \right) =$$

$$= \frac{1}{2}\int \frac{dz}{\sqrt{z}} = \sqrt{z} = \sqrt{t^2+k^2} \Rightarrow$$

$$= t\sqrt{t^2+k^2} - \underbrace{\int \sqrt{t^2+k^2}\,dt}_{I_1} \Rightarrow$$

$$I_1 = t\sqrt{t^2+k^2} - I_1 + k^2 I_3 \Rightarrow I_1 = \frac{t}{2}\sqrt{t^2+k^2} + \frac{k^2}{2} I_3$$

$$I = \sqrt{3}\left(\frac{t}{2}\sqrt{t^2+k^2} + \frac{k^2}{2} I_3 \right) \Rightarrow$$

$$I_3 = \int \frac{dt}{\sqrt{t^2+k^2}} = \ln\left(t + \sqrt{t^2+k^2}\right) \Rightarrow$$

$$I = \sqrt{3}\left[\frac{t}{2}\sqrt{t^2+k^2} + \frac{k^2}{2}\ln\left(t + \sqrt{t^2+k^2}\right) \right] \Rightarrow$$

$$I = \sqrt{3}\left[\frac{3x+5}{6}\sqrt{x^2+\frac{10}{3}x+3} + \frac{1}{9}\ln\left(\frac{3x+5}{3} + \sqrt{x^2+\frac{10}{3}x+3} \right) \right] + C$$

ili

$$I = \frac{3x+5}{6}\sqrt{3x^2+10x+9} + \frac{\sqrt{3}}{9}\ln\left(3x+5+\sqrt{3}\sqrt{3x^2+10x+9} \right) + C'$$

gdje je: $C' = -\frac{\sqrt{3}}{9}\ln 3 + C$.

(2) $$I = \int \frac{4x-3}{5-7x}\,dx = -\frac{1}{7}\int \frac{4x-3}{x-\frac{5}{7}}\,dx \Rightarrow$$

$$\left| (4x+3) : \left(x - \frac{5}{7}\right) = 4 + \frac{41}{7}\cdot\frac{1}{x-\frac{5}{7}} \right| \Rightarrow$$

$$I = -\frac{1}{7}\int\left(4 + \frac{41}{7}\cdot\frac{1}{x-\frac{5}{7}} \right)dx = -\frac{1}{7}\left(4x + \frac{41}{7}\ln\left| x - \frac{5}{7} \right| \right) + C.$$

(Z) $I = \int \frac{x^3-2x^2+4}{x^3(x-2)^2}\,dx \Rightarrow \quad x^3(x-2)^2=0, \quad \left.\begin{matrix} x^3=0 \\ (x-2)^2=0 \end{matrix}\right\} \begin{matrix} x_{1,2,3}=0 \\ x_{4,5}=2. \end{matrix}$

$$\Rightarrow \quad \frac{x^3-2x^2+4}{x^3(x-2)^2} \equiv \frac{A}{x^3}+\frac{B}{x^2}+\frac{C}{x}+\frac{D}{(x-2)^2}+\frac{E}{x-2}$$

$$x^3-2x^2+4 \equiv A(x-2)^2+Bx(x-2)^2+Cx^2(x-2)^2+Dx^3+Ex^3(x-2).$$

$$x^3-2x^2+4 \equiv x^4(C+E)+x^3(B-4C+D-2E)+x^2(A-4B+4C)+$$
$$+x(-4A+4B)+4A \quad \Rightarrow$$

$$\begin{matrix} C+E=0 \\ B-4C+D-2E=1 \\ A-4B+4C=-2 \\ -4A+4B=0 \\ 4A=4 \end{matrix} \quad \Rightarrow \quad \begin{matrix} E=-\frac{1}{4} \\ D=\frac{1}{2} \\ C=\frac{1}{4} \\ B=1 \\ A=1 \end{matrix}$$

$$\frac{x^3-2x^2+4}{x^3(x-2)^2} = \frac{1}{x^3}+\frac{1}{x^2}+\frac{1}{4}\cdot\frac{1}{x}+\frac{1}{2}\frac{1}{(x-2)^2}-\frac{1}{4}\cdot\frac{1}{x-2}$$

$$I = \int\frac{1}{x^3}\,dx+\int\frac{1}{x^2}\,dx+\frac{1}{4}\int\frac{1}{x}\,dx+\frac{1}{2}\int\frac{dx}{(x-2)^2}-\frac{1}{4}\int\frac{dx}{x-2}=$$

$$= -\frac{1}{2}\cdot\frac{1}{x^2}-\frac{1}{x}+\frac{1}{4}\ln|x|-\frac{1}{2}\cdot\frac{1}{x-2}-\frac{1}{4}\ln|x-2|+C=$$

$$= \frac{1}{4}\ln\left|\frac{x}{x-2}\right|-\frac{1}{x}\left(1+\frac{1}{2x}\right)-\frac{1}{2(x-2)}+C.$$

(Z) $\int \frac{x^3+4x^2-2x+1}{x^4+x}\,dx$

$x^4+x=0, \quad x(x^3+1)=0 \Rightarrow x(x+1)(x^2-x+1)=0$

$x_1=0, \quad x_2=1, \quad x_{3,4}=\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$

$$\frac{x^3+4x^2-2x+1}{x(x+1)(x^2-x+1)} \equiv \frac{A}{x}+\frac{B}{x+1}+\frac{kx+L}{x^2-x+1} \quad \Big| \cdot x(x+1)(x^2-x+1)$$

$$x^3+4x^2-2x+1 \equiv A(x^3+1)+Bx(x^2-x+1)+(kx+L)x(x+1)$$

$$x^3+4x^2-2x+1 \equiv (A+B+k)x^3+(-B+k+L)x^2+(B+L)x+A$$

$A+B+k=1$ $\qquad B=-2$

$-B+k+L=4$ $\qquad k=2$

$B+L=-2$ $\qquad L=0$

$A=1.$

$$I=\int\frac{dx}{x}-2\int\frac{dx}{x+1}+2\underbrace{\int\frac{x\,dx}{x^2-x+1}}_{I_1}=\ln\frac{|x|}{(x+1)^2}+2I_1$$

$$I_1=\int\frac{x\,dx}{x^2-x+1}\Rightarrow\left|\begin{matrix}x^2-x+1=t\\(2x-1)dx=dt\end{matrix}\right|\Rightarrow$$ pišemo x u brojniku

kao izraz: $\quad x=\frac{2x-1}{2}+\frac{1}{2}$

$$I_1=\frac{1}{2}\int\frac{(2x-1)dx}{x^2-x+1}+\frac{1}{2}\underbrace{\int\frac{dx}{x^2-x+1}}_{I_2}=\frac{1}{2}\int\frac{dt}{t}+\frac{1}{2}I_2=\frac{1}{2}\ln|t|+$$

$$+\frac{1}{2}I_2.$$

$$I_1=\frac{1}{2}\ln(x^2-x+1)+\frac{1}{2}I_2.$$

$$I_2=\int\frac{dx}{x^2-x+1}=\text{prema predtipu } A=\int\frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}=$$

$$=\left(x-\frac{1}{2}=t,\ dx=dt,\ \frac{3}{4}=k^2\right)=\int\frac{dt}{t^2+k^2}=\frac{1}{k}\operatorname{arctg}\frac{t}{k}=$$

$$=\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$$

$$I_1=\frac{1}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}$$

$$I=\ln\frac{|x|(x^2-x+1)}{(x+1)^2}+\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+C.$$

(Z) $\quad I=\int\frac{dx}{(x^2+a^2)^2}=\frac{1}{a^2}\int\frac{(a^2+x^2)-x^2}{(x^2+a^2)^2}\,dx=$

$$=\frac{1}{a^2}\left[\int\frac{dx}{x^2+a^2}-\int\frac{x^2dx}{(x^2+a^2)^2}\right]=\frac{1}{a^2}\left[\frac{1}{a}\operatorname{arctg}\frac{x}{a}-I_1\right].$$

$$I_1=\int\frac{x^2dx}{(x^2+a^2)^2}=\left[x=u,\ dx=du,\ dv=\frac{x\,dx}{(x^2+a^2)^2},\ v=\int\frac{x\,dx}{(x^2+a^2)^2}\right.$$

$$= \left(x^2+a^2=z\,;\; x\,dx = \tfrac{1}{2}dz \right) = \tfrac{1}{2}\int \frac{dz}{z^2} = -\tfrac{1}{2}\,\frac{1}{x^2+a^2} \Big] =$$

$$= -\frac{x}{2}\cdot\frac{1}{x^2+a^2} + \frac{1}{2}\int\frac{dx}{x^2+a^2} = -\frac{x}{2(x^2+a^2)} + \frac{1}{2a}\,\mathrm{arctg}\,\frac{x}{a}.$$

$$I = \frac{1}{a^2}\left[\frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a} + \frac{x}{2(x^2+a^2)} - \frac{1}{2a}\,\mathrm{arctg}\,\frac{x}{a}\right] =$$

$$= \frac{1}{2a^2}\left(\frac{x}{x^2+a^2} + \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a}\right) + C.$$

(Z) $\displaystyle\int\frac{dx}{(x+1)\sqrt{1-x}} \Rightarrow \sqrt{1-x} = t$

$1-x = t^2, \quad x = 1-t^2$

$dx = -2t\,dt$

$x+1 = 1-t^2+1 = -(t^2-2)$

$$I = 2\int\frac{t\,dt}{(t^2-2)t} = 2\int\frac{dt}{t^2-2} = 2\,\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| =$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{1-x}-\sqrt{2}}{\sqrt{1-x}+\sqrt{2}}\right| + C.$$

(Z) $\displaystyle\int\frac{1}{(1-x)^2}\sqrt{\frac{1-x}{1+x}}\,dx = \left|\sqrt{\frac{1-x}{1+x}} = t, \quad \frac{1-x}{1+x} = t^2 \Rightarrow x = \frac{1-t^2}{1+t^2},\right.$

$$\left. dx = -\frac{4t\,dt}{(1+t^2)^2}, \quad 1-x = \frac{2t^2}{1+t^2}\right| = -4\int\frac{(1+t^2)^2}{4t^4}\cdot\frac{t^2\,dt}{(1+t^2)^2} =$$

$$= -\int\frac{dt}{t^2} = \frac{1}{t} = \sqrt{\frac{1+x}{1-x}} + C.$$

(Z) $\displaystyle\int\frac{dx}{x(\sqrt{x}+\sqrt[5]{x^2})} \Rightarrow \left.\begin{array}{l}\sqrt{x} = x^{\frac{1}{2}}\\ \sqrt[5]{x^2} = x^{\frac{2}{5}}\end{array}\right\}\; x = t^{10}, \; dx = 10t^9\,dt$

$t = \sqrt[10]{x}.$

$$I = 10\int\frac{t^9\,dt}{t^{10}(t^5+t^4)} = 10\int\frac{dt}{t^5(t+1)} \Rightarrow$$

$$\frac{1}{t^5(t+1)} = \frac{1}{t^5} - \frac{1}{t^4} + \frac{1}{t^3} - \frac{1}{t^2} + \frac{1}{t} - \frac{1}{t+1}.$$

$$I = -\frac{10}{4t^4} + \frac{10}{3t^3} - \frac{10}{2t^2} + \frac{10}{t} + 10\ln|t| - 10\ln|t+1| =$$

$$= -\frac{5}{2\sqrt[5]{x^2}} + \frac{10}{3\sqrt[10]{x^3}} - \frac{5}{\sqrt[5]{x}} + \frac{10}{\sqrt[10]{x}} + \ln\frac{|x|}{(1+\sqrt[10]{x})^{10}} + C.$$

(Z) $\displaystyle\int \frac{dx}{x-\sqrt{x^2-x+1}} \Rightarrow a=1>0 \Rightarrow \sqrt{a}=\sqrt{1}=1 \Rightarrow$

$$\sqrt{x^2-x+1} = 1\cdot x + t \quad /^2$$

$$x^2-x+1 = x^2+2tx+t^2$$

$$x = -\frac{t^2-1}{2t+1}$$

$$dx = -\frac{2(t^2+t+1)}{(2t+1)^2}\,dt$$

$$\sqrt{x^2-x+1} = \frac{t^2+t+1}{2t+1}$$

$$x-\sqrt{x^2-x+1} = \frac{2t^2+1}{2t+1}$$

$$I = \frac{1}{2}\int \frac{t^2+t+1}{t\left(t+\frac{1}{2}\right)^2}\,dt$$

$$\frac{t^2+t+1}{t\left(t+\frac{1}{2}\right)^2} \equiv \frac{A}{t} + \frac{B}{\left(t+\frac{1}{2}\right)^2} + \frac{C}{t+\frac{1}{2}} \Rightarrow A=4,\ B=-\frac{3}{2},\ C=-3.$$

$$I = \frac{1}{2}\left(4\int\frac{dt}{t} - \frac{3}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2} - 3\int\frac{dt}{t+\frac{1}{2}}\right) =$$

$$= 2\ln|t| + \frac{3}{2t+1} - \frac{3}{2}\ln(2t+1) + \underbrace{\frac{3}{2}\ln 2 + C'}_{C}$$

$$\Rightarrow t = \sqrt{x^2-x+1} - x \Rightarrow$$

$$I = 2\ln\left|\sqrt{x^2-x+1}-x\right| + \frac{3}{2\sqrt{x^2-x+1}-2x+1} - \frac{3}{2}\ln\left|2\sqrt{x^2-x+1}-2x+1\right| + C.$$

(Z) $\displaystyle\int \frac{dx}{3x+\sqrt{5+8x-4x^2}}$, $a=-4<0 \Rightarrow$

$$5+8x-4x^2 = -4\left(x-\frac{5}{2}\right)\left(x+\frac{1}{2}\right) = 4\left(\frac{5}{2}-x\right)\left(x+\frac{1}{2}\right)$$

$$\sqrt{5+8x-4x^2} = \sqrt{4\left(\frac{5}{2}-x\right)\left(x+\frac{1}{2}\right)} = t\left(\frac{5}{2}-x\right) \quad /^2$$

$$4\left(\frac{5}{2}-x\right)\left(x+\frac{1}{2}\right) = t^2\left(\frac{5}{2}-x\right)^2 \Rightarrow$$

$$4\left(x+\frac{1}{2}\right) = t^2\left(\frac{5}{2}-x\right) \Rightarrow x = \frac{1}{2}\,\frac{5t^2-4}{t^2+4}$$

$$dx = \frac{24t\,dt}{(t^2+4)^2}$$

$$\sqrt{5+8x-4x^2} = t\left(\frac{5}{2}-x\right) = t\left(\frac{5}{2}-\frac{5t^2-4}{2(t^2+4)}\right) = \frac{12t}{t^2+4}$$

$$3x+\sqrt{5+8x-4x^2} = \frac{3}{2}\,\frac{5t^2-4}{t^2+4} + \frac{12t}{t^2+4} = \frac{3}{2}\,\frac{5t^2+8t-4}{t^2+4}$$

$$I = 16\int \frac{t\,dt}{(5t^2+8t-4)(t^2+4)} = \frac{16}{5}\int \frac{t\,dt}{\left(t-\frac{2}{5}\right)(t+2)(t^2+4)} \Rightarrow$$

nakon rastavljanja u parcijalne razlomke:

$$I = \frac{1}{39}\left(5\ln|5t-2| + 13\ln|t+2| - \frac{125}{7}\ln(t^2+4) + 12\,\mathrm{arctg}\,\frac{t}{2}\right) + C$$

gdje je: $t = \dfrac{2\sqrt{5+8x-4x^2}}{5-2x}$.

(2) $$\int \frac{dx}{(x^2+16)\sqrt{9-x^2}} \Rightarrow \quad x = 3\sin t,\ dx = 3\cos t\,dt,$$

$$\sqrt{9-x^2} = 3\sqrt{1-\sin^2 t} = 3\cos t.$$

$$I = \int \frac{3\cos t\,dt}{(9\sin^2 t+16)\cdot 3\cos t} = \int \frac{dt}{9\sin^2 t + 16(\sin^2 t+\cos^2 t)} =$$

$$= \int \frac{dt}{25\sin^2 t + 16\cos^2 t} = \int \frac{dt}{\cos^2 t(25\,\mathrm{tg}^2 t + 16)} = \left(\mathrm{tg}\,t = z,\right.$$

$$\left.\frac{dt}{\cos^2 t} = dz\right) = \int \frac{dz}{25z^2+16} = \frac{1}{25}\int \frac{dz}{z^2+\left(\frac{4}{5}\right)^2} =$$

$$= \frac{1}{25}\cdot\frac{5}{4}\,\mathrm{arctg}\,\frac{5}{4}z = \frac{1}{20}\,\mathrm{arctg}\left(\frac{5}{4}\,\mathrm{tg}\,t\right) + C.$$

$$\mathrm{tg}\,t = \frac{\sin t}{\cos t} = \frac{\sin t}{\sqrt{1-\sin^2 t}} = \left(\text{prema } x = 3\sin t,\ \sin t = \frac{x}{3}\right) =$$

$$= \frac{x}{3\sqrt{1-\frac{x^2}{9}}} = \frac{x}{\sqrt{9-x^2}},\ \text{pa je } I = \frac{1}{20}\,\mathrm{arctg}\,\frac{5x}{4\sqrt{9-x^2}} + C.$$

(2) $\int \frac{x^2-x+1}{(x^2+1)\sqrt{x^2+1}}\,dx \Rightarrow \quad x=\mathrm{ctg}\,t,\ dx=-\frac{dt}{\sin^2 t},$

$$x^2+1=1+\mathrm{ctg}^2 t=\frac{1}{\sin^2 t}$$

$$\sqrt{x^2+1}=\frac{1}{\sin t}$$

$$I=-\int\frac{(\mathrm{ctg}^2 t-\mathrm{ctg}\,t+1)\sin^2 t\sin t\,dt}{\sin^2 t}=-\int\left(\frac{\cos^2 t}{\sin t}-\cos t+\sin t\right)dt=$$

$$=-\int\left(\frac{1}{\sin t}-\sin t-\cos t+\sin t\right)dt=-\int\frac{dt}{\sin t}+\int\cos t\,dt=$$

$$=-\ln\left|\mathrm{tg}\frac{t}{2}\right|+\sin t=\ln\left|\mathrm{ctg}\frac{t}{2}\right|+\sin t+C$$

Vratimo se na x: $\sin t=\frac{1}{\sqrt{1+\mathrm{ctg}^2 t}}=\frac{1}{\sqrt{1+x^2}}$

$$\cos t=\frac{x}{\sqrt{1+x^2}},\quad \mathrm{ctg}\frac{t}{2}=\sqrt{\frac{1+\cos t}{1-\cos t}}=$$

$$=\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}=\sqrt{\frac{(\sqrt{1+x^2}+x)^2}{1+x^2-x^2}}=x+\sqrt{1+x^2}$$

$$I=\ln\left|x+\sqrt{1+x^2}\right|+\frac{1}{\sqrt{1+x^2}}+C.$$

ili jednostavnije ako podijelimo polinom u brojniku s polinomom u nazivniku.

$$\int\frac{x^2-x+1}{(x^2+1)\sqrt{x^2+1}}\,dx=\int\left(1-\frac{x}{x^2+1}\right)\frac{dx}{\sqrt{x^2+1}}=\int\frac{dx}{\sqrt{x^2+1}}-\int\frac{x\,dx}{\sqrt{(x^2+1)^3}}=$$

$$=\left(x^2+1=t,\ x\,dx=\frac{dt}{2}\right)=\ln\left|x+\sqrt{1+x^2}\right|-\frac{1}{2}\int\frac{dt}{t^{3/2}}=$$

$$=\ln\left|x+\sqrt{1+x^2}\right|+\frac{1}{\sqrt{1+x^2}}+C.$$

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(Z) $\displaystyle\int \frac{3x^3-8x+5}{\sqrt{x^2-4x-7}}\,dx \equiv (A_2x^2+A_1x+A_0)\sqrt{x^2-4x-7}+A_3\int\frac{dx}{\sqrt{x^2-4x-7}}$

$$\frac{3x^3-8x+5}{\sqrt{x^2-4x-7}} \equiv (A_2x^2+A_1x+A_0)\frac{x-2}{\sqrt{x^2-4x-7}}+\sqrt{x^2-4x-7}\,(2A_2x+A_1)+\frac{A_3}{\sqrt{x^2-4x-7}}$$

$$3x^3-8x+5 \equiv 3A_2x^3+(2A_1-10A_2)x^2+(-14A_2-6A_1+A_0)x+(A_3-7A_1-2A_0)$$

$3A_2=3 \Rightarrow A_2=1.$

$2A_1-10=0 \Rightarrow A_1=5.$

$-14-30+A_0=-8 \Rightarrow A_0=36.$

$A_3-35-72=5 \Rightarrow A_3=112.$

$$I=(x^2+5x+36)\sqrt{x^2-4x-7}+112\underbrace{\int\frac{dx}{\sqrt{x^2-4x-7}}}_{I_1}$$

$$I_1=\int\frac{dx}{\sqrt{x^2-4x-7}}=\int\frac{dx}{\sqrt{(x-2)^2-11}}=(x-2=t,\ dx=dt,\ 11=k^2)=$$

$$=\int\frac{dx}{\sqrt{t^2-k^2}}=\ln\left|t+\sqrt{t^2+k^2}\right|=\ln\left|x-2+\sqrt{x^2-4x-7}\right|$$

$$I=(x^2+5x+36)\sqrt{x^2-4x-7}+112\ln\left|x-2+\sqrt{x^2-4x-7}\right|+C.$$

(Z) $\displaystyle\int(2x-5)\sqrt{2+3x-x^2}\,dx=\int\frac{(2x-5)(2+3x-x^2)}{\sqrt{2+3x-x^2}}\,dx=$

$$=\int\frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}}\,dx\equiv(A_2x^2+A_1x+A_0)\sqrt{2+3x-x^2}+A_3\int\frac{dx}{\sqrt{2+3x-x^2}}$$

$$\frac{-2x^3+11x^2-11x-10}{\sqrt{2+3x-x^2}}\equiv(A_2x^2+A_1x+A_0)\frac{3-2x}{2\sqrt{2+3x-x^2}}+\sqrt{2+3x-x^2}\,(2A_2x+A_1)+$$

$$+\frac{A_3}{\sqrt{2+3x-x^2}}$$

$$-4x^3+22x^2-22x-20\equiv-6A_2x^3+(15A_2-4A_1)x^2+(8A_2+3A_1-2A_0)x+(4A_1+2A_0)$$

$A_2=\frac{2}{3},\ A_1=-3,\ A_0=\frac{1}{6},\ A_3=-4$

$$I=\left(\frac{2}{3}x^2-3x+\frac{1}{6}\right)\sqrt{2+3x-x^2}-4\int\frac{dx}{\sqrt{2+3x-x^2}}=$$

$$=\left(\frac{2}{3}x^2-3x+\frac{1}{6}\right)\sqrt{2+3x-x^2}-4\arcsin\frac{2x-3}{\sqrt{17}}+C.$$

(2) $$\int \frac{dx}{(x-1)\sqrt{x^2+x+1}} = \left(x-1=\frac{1}{t},\ x=\frac{1}{t}+1=\frac{1+t}{t},\ dx=-\frac{dt}{t^2} \right) =$$

$$= -\int \frac{dt}{t^2\cdot\frac{1}{t}\sqrt{\frac{(1+t)^2}{t^2}+\frac{1+t}{t}+1}} = -\int \frac{dt}{\sqrt{3t^2+3t+1}} =$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}} = \left(t+\frac{1}{2}=z,\ dt=dz,\ \frac{1}{12}=k^2 \right) = -\frac{1}{\sqrt{3}}\int \frac{dz}{\sqrt{z^2+k^2}} =$$

$$= -\frac{1}{\sqrt{3}}\ln\left|z+\sqrt{k^2+z^2}\right| = -\frac{1}{\sqrt{3}}\ln\left|t+\frac{1}{2}+\sqrt{t^2+t+\frac{1}{3}}\right| =$$

$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{2\sqrt{3}t+\sqrt{3}+2\sqrt{3t^2+3t+1}}{2\sqrt{3}}\right| = -\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}x+\sqrt{3}+2\sqrt{x^2+x+1}}{x-1}\right| + C.$$

(2) $$\int \frac{(3x-5)\,dx}{(x-1)^2\sqrt{x^2+4}} \Rightarrow \frac{3x-5}{(x-1)^2} = \frac{A}{(x-1)^2}+\frac{B}{x-1} \quad /\cdot(x-1)^2$$

$$3x-5 \equiv A+Bx-B \Rightarrow B=3;\ A=-2$$

$$\frac{3x-5}{(x-1)^2} = -\frac{2}{(x-1)^2}+\frac{3}{x-1}$$

$$I = -2\underbrace{\int \frac{dx}{(x-1)^2\sqrt{x^2+4}}}_{I_1} + 3\underbrace{\int \frac{dx}{(x-1)\sqrt{x^2+4}}}_{I_2} = -2I_1+3I_2$$

$$I_1 = \int \frac{dx}{(x-1)^2\sqrt{x^2+4}} = \left(x-1=\frac{1}{t},\ x=\frac{1}{t}+1,\ dx=-\frac{dt}{t^2},\ t=\frac{1}{x-1} \right) =$$

$$= \left(\sqrt{x^2+4} = \frac{1}{t}\sqrt{1+2t+5t^2} \right) =$$

$$= -\int \frac{t\,dt}{\sqrt{1+2t+5t^2}} = -I_1^{\circ}$$

$$I_1^{\circ} = \int \frac{t\,dt}{\sqrt{1+2t+5t^2}} = A_0\sqrt{5t^2+2t+1} + A_1\int \frac{dt}{\sqrt{5t^2+2t+1}}$$

$$A_0 = \frac{1}{5},\quad A_1 = -\frac{1}{5}$$

$$I_1 = -I_1^{\circ} = -\frac{1}{5}\sqrt{5t^2+2t+1} + \frac{1}{5}\int \frac{dt}{\sqrt{5t^2+2t+1}} = -\frac{1}{5}\sqrt{5t^2+2t+1} +$$

$$+ \frac{1}{5\sqrt{5}}\ln\left|\frac{5t+1+\sqrt{5}\sqrt{5t^2+2t+1}}{5}\right|, \quad t=\frac{1}{x-1} \Rightarrow$$

$$I_1 = -\frac{1}{5}\frac{\sqrt{x^2+4}}{x-1} + \frac{1}{5\sqrt{5}}\ln\left|\frac{x+4+\sqrt{5}\sqrt{x^2+4}}{x-1}\right|.$$

$$I_2 = \int \frac{dx}{(x-1)\sqrt{x^2+4}} = \left(x-1 = \frac{1}{t},\ dx = -\frac{dt}{t^2},\ t = \frac{1}{x-1},\ x = \frac{t+1}{t} \right) =$$

$$= -\frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 + \frac{2}{5}t + \frac{1}{5}}} = -\frac{1}{\sqrt{5}} \ln \left| \frac{5t+1+\sqrt{5}\sqrt{5t^2+2t+1}}{5} \right|,$$

$$t = \frac{1}{x-1}, \quad I_2 = -\frac{1}{\sqrt{5}} \ln \left| \frac{x+4+\sqrt{5}\sqrt{x^2+4}}{x-1} \right|$$

$$I = -2I_1 + 3I_2 = \frac{2}{5} \frac{\sqrt{x^2+4}}{x-1} - \frac{17}{5\sqrt{5}} \ln \left| \frac{x+4+\sqrt{5}\sqrt{x^2+4}}{x-1} \right| + C.$$

(Z) $\int x^5 \sqrt[3]{(1+x^3)^2}\, dx = \int x^5 (1+x^3)^{\frac{2}{3}} dx \Rightarrow p = 5, q = 3, r = \frac{2}{3}, s = 3.$

$\frac{p+1}{q} = \frac{5+1}{3} = 2 \Rightarrow$ cio broj $\Rightarrow$ $1+x^3 = t^3 \Rightarrow x = \sqrt[3]{t^3-1}$

$t = \sqrt[3]{1+x^3}$

$dx = \frac{t^2\, dt}{\sqrt[3]{(t^3-1)^2}}$

$$I = \int \sqrt[3]{(t^3-1)^5} \cdot \sqrt[3]{t^6} \cdot \frac{t^2\, dt}{\sqrt[3]{(t^3-1)^2}} = \int (t^3-1) t^4\, dt = \frac{t^8}{8} - \frac{t^5}{5}$$

$$t = \sqrt[3]{1+x^3} \Rightarrow I = \frac{1}{8} \sqrt[3]{(1+x^3)^8} - \frac{1}{5} \sqrt[3]{(1+x^3)^5} + C$$

(Z) $\int \frac{dx}{x^2 \sqrt[3]{(1+x^3)^5}} = \int x^{-2} (1+x^3)^{-\frac{5}{3}} dx.$

$p = -2, q = 3, r = -\frac{5}{3}, s = 3$

$\frac{p+1}{q} = \frac{-2+1}{3} = -\frac{1}{3} \neq$ cio broj

$\frac{p+1}{q} + r = -\frac{1}{3} - \frac{5}{3} = -2 =$ cio broj $\Rightarrow 1+x^3 = t^3 x^3 \Rightarrow$

$t = \frac{\sqrt[3]{1+x^3}}{x}, \quad 1 = x^3(t^3-1), \quad x^3 = \frac{1}{t^3-1}, \quad 1+x^3 = \frac{t^3}{t^3-1}$

$x = \frac{1}{\sqrt[3]{t^3-1}} = (t^3-1)^{-\frac{1}{3}}, \quad dx = -t^2 (t^3-1)^{-\frac{4}{3}} dt$

$$I = -\int t^{-3}(t^3-1)\, dt = -\int dt + \int \frac{dt}{t^3} = -t - \frac{1}{2t^2} = -\frac{1+2t^3}{2t^2}$$

$$t = \frac{\sqrt[3]{1+x^3}}{x} \Rightarrow I = -\frac{3x^3+2}{2x\sqrt[3]{(1+x^3)^2}} + C.$$

(Z) $\int x^5 \sqrt[7]{2-3x^6}\,dx = \left| 2-3x^6 = t \right| \Rightarrow dt = -18x^5 dx, \quad x^5 dx = -\frac{dt}{18}$

$I = -\frac{1}{18}\int \sqrt[7]{t}\,dt = -\frac{7}{144}\sqrt[7]{t^8} = -\frac{7}{144}\sqrt[7]{(2-3x^6)^8} + C.$

(Z) $\int \frac{dx}{\sin x} = \left| \sin x = \frac{2t}{1+t^2}, \ \cos x = \frac{1-t^2}{1+t^2}, \ dx = \frac{2dt}{1+t^2}, \ \operatorname{tg}\frac{x}{2} = t \right| =$

$= \int \frac{2dt(1+t^2)}{(1+t^2)\,2t} = \int \frac{dt}{t} = \ln|t| = \ln\left|\operatorname{tg}\frac{x}{2}\right| + C.$

(Z) $\int \frac{dx}{\cos x} = \int \frac{dx}{\sin(x+\frac{\pi}{2})} = \left(x+\frac{\pi}{2} = t, \ dx = dt\right) = \int \frac{dt}{\sin t} =$

$= \ln\left|\operatorname{tg}\frac{t}{2}\right| = \ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C.$

(Z) $\int \frac{2-\sin x}{2+\cos x}\,dx = \left|\operatorname{tg}\frac{x}{2} = t\right| = \int \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} =$

$= 4\int \frac{t^2-t+1}{(1+t^2)(t^2+3)}\,dt \Rightarrow \frac{t^2-t+1}{(t^2+1)(t^2+3)} \equiv \frac{At+B}{t^2+1} + \frac{Ct+D}{t^2+3}$

$A = -\frac{1}{2}, \ C = \frac{1}{2}, \ B = 0, \ D = 1.$

$I = -\int \frac{2t\,dt}{t^2+1} + \int \frac{2t+4}{t^2+3}\,dt = -\ln(t^2+1) + \ln(t^2+3) + 4\int \frac{dt}{t^2+3} =$

$= \ln\left(\frac{t^2+3}{t^2+1}\right) + \frac{4}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}} = \ln\left(1+\frac{2}{t^2+1}\right) + \frac{4}{\sqrt{3}}\operatorname{arctg}\frac{t}{\sqrt{3}} =$

$= \ln\left(1+\frac{2}{\operatorname{tg}^2\frac{x}{2}+1}\right) + \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right) = \ln\left(1+2\cos^2\frac{x}{2}\right) +$

$+ \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right) = \ln(2+\cos x) + \frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{1}{\sqrt{3}}\operatorname{tg}\frac{x}{2}\right) + C.$

(Z) $\int \frac{dx}{4-3\cos^2 x+5\sin^2 x} \Rightarrow$ integrant je parna f-ja, jer je

$R(-\sin x, -\cos x) = \frac{1}{4-3\cos^2 x+5\sin^2 x} =$

$= R(\sin x, \cos x) \Rightarrow$

$\Rightarrow$ dijelimo integrant sa $\cos^2 x$ i ide smjena $\operatorname{tg} x = t$,

$\frac{dx}{\cos^2 x} = dt, \quad \frac{1}{\cos^2 x} = 1+\operatorname{tg}^2 x = 1+t^2 \Rightarrow$

$$I=\int \frac{\frac{dx}{\cos^2 x}}{\frac{4}{\cos^2 x}-3+5\operatorname{tg}^2 x} = \int \frac{dt}{4(1+t^2)-3+5t^2} = \int \frac{dt}{9t^2+1} =$$

$$= \frac{1}{3}\operatorname{arctg}(3t) = \frac{1}{3}\operatorname{arctg}(3\operatorname{tg}x)+C.$$

(Z) $\int \frac{\sin^3 x}{1+\cos^3 x}\,dx \Rightarrow$ интегрант непаран у $\sin x$. $\Rightarrow \cos x = t$, $\sin x\,dx=-dt$.

$$I=\int \frac{(1-\cos^2 x)\sin x}{1+\cos^2 x}\,dx = -\int \frac{(1-t^2)\,dt}{1+t^2} = \int\left(1-\frac{2}{t^2+1}\right)dt =$$

$$= t-2\operatorname{arctg} t = \cos x - \operatorname{arctg}(\cos x)+C.$$

(Z) $\int \frac{\cos^3 x\,dx}{4\sin^2 x-1} \Rightarrow$ интегрант непаран и $\cos x \Rightarrow \sin x = t$, $dt=\cos x\,dx$.

$$I=\int \frac{\cos^2 x\cdot\cos x}{4\sin^2 x-1}\,dx = \int\frac{(1-\sin^2 x)\cos x\,dx}{4\sin^2 x-1} = \int\frac{(1-t^2)\,dt}{4t^2-1} =$$

$$= -\frac{1}{4}\int\frac{t^2-1}{t^2-\frac{1}{4}}\,dt = -\frac{1}{4}\int\left(1-\frac{3}{4}\,\frac{1}{t^2-\frac{1}{4}}\right)dt = -\frac{t}{4}+\frac{3}{16}\int\frac{dt}{t^2-\frac{1}{4}} =$$

$$= -\frac{t}{4}+\frac{3}{16}\ln\left|\frac{t-\frac{1}{2}}{t+\frac{1}{2}}\right| = -\frac{1}{4}\sin x+\frac{3}{16}\ln\left|\frac{2\sin x-1}{2\sin x+1}\right|+C.$$

(Z) $$\int\frac{dx}{\operatorname{sh}x} = \left|\operatorname{th}\frac{x}{2}=t,\ \operatorname{sh}x=\frac{2t}{1-t^2},\ \operatorname{ch}x=\frac{1+t^2}{1-t^2},\ dx=\frac{2dt}{1-t^2}\right| =$$

$$= \int\frac{2dt\,(1-t^2)}{2t\cdot(1-t^2)} = \int\frac{dt}{t} = \ln\left|\operatorname{th}\frac{x}{2}\right|+C.$$

(Z) $$\int\frac{dx}{\operatorname{ch}x} = 2\int\frac{dx}{e^x+e^{-x}} = \left|e^x=t,\ e^{-x}=\frac{1}{t},\ e^x dx=dt,\ dx=\frac{dt}{e^x}=\frac{dt}{t}\right| =$$

$$= 2\int\frac{dt}{t\left(t+\frac{1}{t}\right)} = 2\int\frac{dt}{t^2+1} = 2\operatorname{arctg}t = 2\operatorname{arctg}e^x+C.$$

(Z) $$\int\frac{dx}{2\operatorname{sh}x+3\operatorname{ch}x} = 2\int\frac{(1-t^2)\,dt}{(1-t^2)(4t+3+3t^2)} = 2\int\frac{dt}{3t^2+4t+3} =$$

$$= \frac{2}{\sqrt{5}}\operatorname{arctg}\frac{3t+2}{\sqrt{5}} = \frac{2}{\sqrt{5}}\operatorname{arctg}\frac{3\operatorname{th}\frac{x}{2}+2}{\sqrt{5}}+C$$

Ⓩ $\int \sin\frac{x}{4}\cdot\cos\frac{x}{3}\,dx = \left|\ \sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\right| = \frac{1}{2}\int\sin\left(\frac{x}{4}+\frac{x}{3}\right)dx +$

$+\frac{1}{2}\int\sin\left(\frac{x}{4}-\frac{x}{3}\right)dx = \frac{1}{2}\int\sin\frac{7}{12}x\,dx + \frac{1}{2}\int\sin\left(-\frac{x}{12}\right)dx =$

$= -\frac{1}{2}\cdot\frac{12}{7}\cos\frac{7}{12}x + \frac{1}{2}\cdot 12\cos\frac{x}{12} = -\frac{6}{7}\cos\frac{7}{12}x + 6\cos\frac{x}{12} + C.$

Ⓩ $\int\cos(ax+b)\cdot\cos(ax-b)\,dx = \left|\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]\right| =$

$= \frac{1}{2}\int\cos 2ax\,dx + \frac{1}{2}\int\cos 2b\,dx = \frac{1}{4a}\sin 2ax + \frac{\cos 2b}{2}x + C.$

Ⓩ $\int\sin x\cdot\sin 2x\cdot\sin 3x\,dx \Rightarrow \left|\sin\alpha\cdot\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]\right| \Rightarrow$

$= \int(\sin x\cdot\sin 2x)\sin 3x\,dx \Rightarrow \left|\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]\right| \Rightarrow$

$= \frac{1}{2}\int(\cos x-\cos 3x)\sin 3x\,dx = \frac{1}{2}\int(\sin 3x\cdot\cos x-\sin 3x\cdot\cos 3x)\,dx =$

$= \frac{1}{4}\int(\sin 4x+\sin 2x-\sin 6x)\,dx = \frac{1}{4}\left(-\frac{1}{2}\cos 2x-\frac{1}{4}\cos 4x+\frac{1}{6}\cos 6x\right) =$

$= -\frac{1}{8}\cos 2x-\frac{1}{16}\cos 4x+\frac{1}{24}\cos 6x = \frac{1}{8}\left(\frac{\cos 6x}{3}-\frac{\cos 4x}{2}-\cos 2x\right)+C.$

Ⓩ $I_2 = \int\sin^2 x\,dx = \int\sin x\cdot\sin x\,dx = (u=\sin x,\ du=\cos x\,dx,\ dv=\sin x\,dx) =$

$= (v=-\cos x) = -\sin x\cos x + \int\cos^2 x\,dx.$

$I_2 = -\sin x\cos x + \int\cos^2 x\,dx = -\sin x\cos x + \int(1-\sin^2 x)\,dx =$

$= -\sin x\cos x + \int dx - \underbrace{\int\sin^2 x\,dx}_{I_2} \Rightarrow$

$2I_2 = -\sin x\cos x + x$

$I_2 = -\frac{\sin x\cos x}{2} + \frac{x}{2}$

$I_2 = \frac{x}{2} - \frac{\sin 2x}{4} + C.$

(Z) $I_4 = \int \sin^4 x\,dx = \int \sin^3 x \cdot \sin x\,dx = (\sin^3 x = u,\ du = 3\sin^2 x\cos x\,dx,$

$= (dv = \sin x\,dx,\ v = -\cos x) = -\sin^3 x\cos x + 3\int \cos^2 x\cdot\sin^2 x\,dx =$

$= -\sin^3 x\cos x + 3\int(1-\sin^2 x)\sin^2 x\,dx.$

$$I_4 = -\sin^3 x\cos x + 3\underbrace{\int\sin^2 x\,dx}_{I_2} - 3\underbrace{\int \sin^4 x\,dx}_{I_4}$$

$$I_4 = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4} I_2$$

$$I_4 = -\frac{1}{4} \sin^3 x\cos x + \frac{3}{4}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right) = -\frac{1}{4}\sin^3 x\cos x + \frac{3}{8} x - \frac{3}{16}\sin 2x + C.$$

(Z) $\int tg^5 x\,dx = \int \frac{t^5}{1+t^2}\,dt = \int\left(t^3 - t + \frac{t}{t^2+1}\right)dt = \frac{t^4}{4} - \frac{t^2}{2} +$

$+\int\frac{t\,dt}{t^2+1} = \left(t^2+1 = z,\ t\,dt = \frac{1}{2}\,dz\right) = \frac{1}{4} t^4 - \frac{1}{2} t^2 + \frac{1}{2}\ln z =$

$= \frac{1}{4} tg^4 x - \frac{1}{2} tg^2 x + \frac{1}{2}\ln(1+tg^2 x) = \frac{1}{4} tg^4 x - \frac{1}{2} tg^2 x - \ln|\cos x| + C.$

$\boxed{1+tg^2 x = \frac{1}{\cos^2 x}}$ smjena: $tg\,x = t$, $x = arc\,tg\,t$, $dx = \frac{dt}{1+t^2}$

(Z) $\int ctg^7 x\,dx = -\int\frac{t^7}{t^2+1}\,dt = -\int[t^7:(t^2+1)]\,dt = -\int\left(t^5 - t^3 + t - \frac{t}{t^2+1}\right)dt =$

$= -\frac{t^6}{6} + \frac{t^4}{4} - \frac{t^2}{2} + \frac{1}{2}\ln(t^2+1) = -\frac{1}{6} ctg^6 x + \frac{1}{4} ctg^4 x - \frac{1}{2} ctg^2 x -$

$- \ln|\sin x| + C.$

(Z) $I_3 = \int\frac{dx}{\sin^3 x} = \int\frac{1}{\sin x}\,\frac{dx}{\sin^2 x} = \left(u = \frac{1}{\sin x},\ du = -\frac{\cos x}{\sin^2 x}\,dx;\right.$

$\left.dv = \frac{dx}{\sin^2 x},\ v = -ctg\,x\right) = -\frac{1}{\sin x}\cdot ctg\,x - \int ctg\,x\cdot\frac{\cos x}{\sin^2 x}\,dx =$

$= -\frac{\cos x}{\sin^2 x} - \int\frac{\cos^2 x}{\sin^3 x}\,dx = -\frac{\cos x}{\sin^2 x} - \int\frac{1-\sin^2 x}{\sin^3 x}\,dx =$

$$= -\frac{\cos x}{\sin^2 x} - \underbrace{\int\frac{dx}{\sin^3 x}}_{I_3} + \underbrace{\int\frac{dx}{\sin x}}_{I_1} \Rightarrow 2I_3 = -\frac{\cos x}{\sin^2 x} + I_1.$$

$$I_3 = -\frac{1}{2}\,\frac{\cos x}{\sin^2 x} + \frac{1}{2} I_1 = -\frac{1}{2}\,\frac{\cos x}{\sin^2 x} + \frac{1}{2}\ln\left|tg\frac{x}{2}\right| + C.$$

(Z) $I_4 = \int \frac{dx}{\sin^4 x} = \int \frac{1}{\sin^2 x}\cdot\frac{dx}{\sin^2 x} = \left| u = \frac{1}{\sin^2 x},\ du = -\frac{2\cos x\, dx}{\sin^3 x}\right|$

$$\left| dv = \frac{dx}{\sin^2 x},\ v = -\operatorname{ctg} x\right| = -\frac{\operatorname{ctg} x}{\sin^2 x} - 2\int \frac{\operatorname{ctg} x \cdot \cos x}{\sin^3 x} =$$

$$= -\frac{\cos x}{\sin^3 x} - 2\int\frac{1-\sin^2 x}{\sin^4 x}\,dx = -\frac{\cos x}{\sin^3 x} - 2\underbrace{\int\frac{dx}{\sin^4 x}}_{I_4} + 2\underbrace{\int \frac{dx}{\sin^2 x}}_{I_2}.$$

$$3I_4 = -\frac{\cos x}{\sin^3 x} + 2I_2 \Rightarrow I_4 = -\frac13\frac{\cos x}{\sin^3 x} + \frac23 I_2 = -\frac13\frac{\cos x}{\sin^3 x} - \frac23 \operatorname{ctg} x + c$$

ili

$$\int\frac{dx}{\sin^4 x} = \int\left(\frac{1}{\sin^2 x}\right)^2 dx = \int(1+\operatorname{ctg}^2 x)^2 dx = \int(1+2\operatorname{ctg}^2 x+\operatorname{ctg}^4 x)\,dx =$$

$$= \int dx + 2\int \operatorname{ctg}^2 x\, dx + \int \operatorname{ctg}^4 x = x - 2\operatorname{ctg} x - 2x - \frac13\operatorname{ctg}^3 x + \operatorname{ctg} x + x =$$

$$= -\operatorname{ctg} x - \frac13\operatorname{ctg}^3 x + c.$$

(Z) $I_5 = \int\frac{dx}{\cos^5 x} = \int \frac{1}{\cos^3 x}\cdot\frac{1}{\cos^2 x}\,dx = \left|u=\frac{1}{\cos^3 x},\ du = 3\frac{\sin x}{\cos^4 x}\,dx\right.$

$$\left|dv = \frac{dx}{\cos^2 x},\ v = \operatorname{tg} x\right| = \frac{1}{\cos^3 x}\operatorname{tg} x - 3\int \operatorname{tg} x\cdot\frac{\sin x\,dx}{\cos^4 x} =$$

$$= \frac{\sin x}{\cos^4 x} - 3\int\frac{1-\cos^2 x}{\cos^5 x}\,dx = \frac{\sin x}{\cos^4 x} - 3\underbrace{\int\frac{dx}{\cos^5 x}}_{I_5} + 3\underbrace{\int\frac{dx}{\cos^3 x}}_{I_3} =$$

$$= \frac{\sin x}{\cos^4 x} - 3I_5 + 3I_3.$$

$$4I_5 = \frac{\sin x}{\cos^4 x} + 3I_3$$

$$I_5 = \frac14\frac{\sin x}{\cos^4 x} + \frac34 I_3, \quad I_3 = \frac12\frac{\sin x}{\cos^2 x} + \frac12\ln\left|\operatorname{tg}\left(\frac x2+\frac\pi4\right)\right|$$

$$I_5 = \frac14 \frac{\sin x}{\cos^4 x} + \frac38\frac{\sin x}{\cos^2 x} + \frac38\ln\left|\operatorname{tg}\left(\frac x2+\frac\pi4\right)\right|+C.$$

(Z) $\int \sin^2 x\cdot\cos^3 x\, dx = (\sin x = t,\ \cos x\, dx = dt) = \int\sin^2 x(1-\sin^2 x)\cos x\, dx =$

$$= \int t^2(1-t^2)\,dt = \int t^2 dt - \int t^4 dt = \frac{t^3}{3}-\frac{t^5}{5} = \frac13\sin^3 x - \frac15\sin^5 x + C.$$

(Z) $\int\frac{\cos^3 x}{\sin^4 x}\,dx = |\sin x = t,\ \cos x\, dx = dt| = \int\frac{(1-\sin^2 x)\cos x\, dx}{\sin^4 x} = \int\frac{(1-t^2)\,dt}{t^4} =$

$$= \int\frac{dt}{t^4} - \int\frac{dt}{t^2} = -\frac13\cdot\frac{1}{t^3} + \frac1t = -\frac13\,\frac{1}{\sin^3 x} + \frac{1}{\sin x} + C.$$

(z) $\int \frac{\sin^4 x}{\cos^2 x}\,dx = \int \frac{(1-\cos^2 x)^2}{\cos^2 x}\,dx = \int \frac{1-2\cos^2 x+\cos^4 x}{\cos^2 x}\,dx =$

$= \int \frac{dx}{\cos^2 x} - 2\int dx + \int \cos^2 x\,dx = \operatorname{tg} x - 2x + \frac{x}{2} + \frac{\sin 2x}{4} =$

$= \operatorname{tg} x + \frac{1}{4}\sin 2x - \frac{3}{2}x + C.$

(z) $\int \sin^5 x\cdot\cos^5 x\,dx = \int \sin^5 x\cdot\cos^4 x\cos x\,dx = \int \sin^5 x(1-\sin^2 x)^2\cos x\,dx =$

$= \left|\sin x = t,\ \cos x\,dx = dt\right| = \int t^5(1-t^2)^2\,dt = \frac{1}{6}t^6 - \frac{1}{4}t^8 + \frac{1}{10}t^{10} =$

$= \frac{1}{6}\sin^6 x - \frac{1}{4}\sin^8 x + \frac{1}{10}\sin^{10} x + C.$

(z) $\int \sin^2 x\cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx = \left(2x=t,\ dx=\frac{1}{2}dt\right) = \frac{1}{8}\int \sin^2 t\,dt =$

$= \frac{1}{8}\left(\frac{t}{2} - \frac{\sin 2t}{4}\right) = \frac{x}{8} - \frac{\sin 4x}{32} + C.$

(z) $\int \frac{dx}{\sin^4 x\cos^4 x} = 16\int \frac{dx}{16\sin^4 x\cos^4 x} = 16\int \frac{dx}{(\sin 2x)^4} = \left|2x=t,\ dx=\frac{1}{2}dt\right| =$

$= 8\int \frac{dt}{\sin^4 t} = 8\left(-\frac{1}{3}\frac{\cos t}{\sin^3 t} - \frac{2}{3}\operatorname{ctg} t\right) = -\frac{8}{3}\frac{\cos 2x}{\sin^3 2x} - \frac{16}{3}\operatorname{ctg} 2x + C =$

$= -8\operatorname{ctg} 2x - \frac{8}{3}\operatorname{ctg}^3 2x + C.$

(z) $\int \sin^2 x\cdot\cos^4 x\,dx = \int (\sin x\cdot\cos x)^2\cos^2 x\,dx = \big|2\sin x\cos x = \sin 2x\ ;$

$;\ 1+\cos 2x = 2\cos^2 x\big| = \frac{1}{8}\int \sin^2 2x(1+\cos 2x)\,dx = \big|2x=t,\ dx=\frac{1}{2}dt,$

$\sin 2x = z,\ 2\cos 2x\,dx = dz\big| = \frac{1}{16}\int \sin^2 t\,dt + \frac{1}{16}\int z^2\,dz =$

$= \frac{1}{16}\left(\frac{t}{2} - \frac{\sin 2t}{4}\right) + \frac{1}{16}\frac{z^3}{3} = \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C.$

(z) $\int \frac{dx}{\sin^2 x\cdot\cos^4 x} = \left|\frac{1}{\sin^2 x} = 1+\operatorname{ctg}^2 x = \frac{\operatorname{tg}^2 x+1}{\operatorname{tg}^2 x},\ \frac{1}{\cos^2 x} = 1+\operatorname{tg}^2 x\right| =$

$= \int \frac{1+\operatorname{tg}^2 x}{\operatorname{tg}^2 x}(1+\operatorname{tg}^2 x)\cdot\frac{dx}{\cos^2 x} = \int \frac{(1+\operatorname{tg}^2 x)^2}{\operatorname{tg}^2 x}\cdot\frac{dx}{\cos^2 x} =$

$= \int\left(\frac{1}{\operatorname{tg}^2 x} + 2 + \operatorname{tg}^2 x\right)\frac{dx}{\cos^2 x} = \left|\operatorname{tg} x = t,\ \frac{dx}{\cos^2 x} = dt\right| = \int \frac{dt}{t^2} +$

$+ 2\int dt + \int t^2\,dt = -\frac{1}{\operatorname{tg} x} + 2\operatorname{tg} x + \frac{1}{3}\operatorname{tg}^3 x + C.$

Ⓩ $\int \frac{dx}{\sin x \cdot \cos^3 x} \quad \Big/ \cdot \sin x = \int \frac{\sin x\, dx}{\sin^2 x \cos^3 x} = \int \frac{\sin x\, dx}{(1-\cos^2 x)\cos^3 x} =$

$= \left| \cos x = t,\ -\sin x\, dx = dt \right| = \int \frac{dt}{t^3(t^2-1)} = -\int \frac{dt}{t^3} - \int \frac{dt}{t} + \frac{1}{2}\int \frac{dt}{t+1} + \frac{1}{2}\int \frac{dt}{t-1} =$

$= \frac{1}{2t^2} - \ln|t| + \frac{1}{2}\ln|t+1| + \frac{1}{2}\ln|t-1| = \frac{1}{2\cos^2 x} + \ln\left|\frac{\sqrt{\cos^2 x - 1}}{\cos x}\right| + C.$

Ⓩ $\int \frac{dx}{\sin x \cdot \sin 2x} = \frac{1}{2}\int \frac{dx}{\sin^2 x \cos x} = \frac{1}{2}\int \frac{\cos x\, dx}{\sin^2 x \cos^2 x} = \frac{1}{2}\int \frac{\cos x\, dx}{\sin^2 x(1-\sin^2 x)} =$

$= \left| \sin x = t,\ \cos x\, dx = dt \right| = \frac{1}{2}\int \frac{dt}{t^2(1+t)(1-t)} = \frac{1}{2}\left(\int \frac{dt}{t^2} + \frac{1}{2}\int \frac{dt}{t+1} + \frac{1}{2}\int \frac{dt}{1-t}\right) =$

$= \frac{1}{2}\left(-\frac{1}{t} + \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|1-t|\right) = -\frac{1}{2\sin x} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C.$

Ⓩ $\int \frac{\sin^3 x}{\sqrt[3]{\cos^4 x}}\, dx \Rightarrow \cos x = t,\ \sin x\, dx = -dt, \Rightarrow \int \frac{(1-\cos^2 x)\sin x\, dx}{\sqrt[3]{\cos^4 x}} =$

$= -\int \frac{(1-t^2)\,dt}{\sqrt[3]{t^4}} = -\int t^{-\frac{4}{3}}dt + \int t^{\frac{2}{3}}dt = -\frac{t^{-\frac{1}{3}}}{-\frac{1}{3}} + \frac{t^{\frac{5}{3}}}{\frac{5}{3}} =$

$= \frac{3}{\sqrt[3]{\cos x}} + \frac{3}{5}\sqrt[3]{\cos^5 x} + C.$

Ⓩ $\int \sqrt[5]{\sin^3 x} \cdot \cos^5 x\, dx \Rightarrow \sin x = t,\ \cos x\, dx = dt \Rightarrow$

$= \int \sqrt[5]{\sin^3 x} \cdot \cos^4 x \cos x\, dx = \int t^{\frac{3}{5}}(1-t^2)^2 dt = \int \left(t^{\frac{3}{5}} - 2t^{\frac{13}{5}} + t^{\frac{23}{5}}\right)dt =$

$= \frac{5}{8}\sqrt[5]{\sin^6 x} - \frac{5}{9}\sqrt[5]{\sin^{18} x} + \frac{5}{28}\sqrt[5]{\sin^{28} x} + C.$

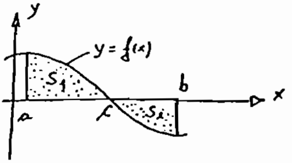
ODREĐENI INTEGRALI

1. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2. $\int_a^a f(x)\,dx = 0$

3. $\int_a^b [f_1(x) \pm f_2(x)]\,dx = \int_a^b f_1(x)\,dx \pm \int_a^b f_2(x)\,dx$

4. $\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx$

5. 

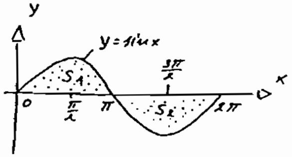
$$S = \int_a^c f(x)\,dx - \int_c^b f(x)\,dx = S_1 + S_2$$

ili

$$S = \int_a^c f(x)\,dx + \int_b^c f(x)\,dx = S_1 + S_2$$

Primjer:

$$\int_0^{2\pi} \sin x\,dx = S_1 - S_2 = 0 \text{ jer je } S_1 = |S_2|$$



6. $\int_0^{2\pi} \sin^{2n+1} x\,dx = 0$, $\int_0^{2\pi} \cos^{2n+1} x\,dx = 0.$

$\int_0^{2\pi} \sin^{2n} x\,dx \neq 0$, $\int_0^{2\pi} \cos^{2n} x\,dx \neq 0.$

7. Njutn - Lajbnicova formula:

$$\int_a^b f(x)\,dx = \Big| F(x) \Big|_a^b = F(b) - F(a)$$

(Z) $\int_0^{2\pi} |\sin^3 x|\,dx = 2\int_0^{\pi} \sin^3 x\,dx = 2\Big| -\frac{1}{3}\sin^2 x\cos x - \frac{2}{3}\cos x \Big|_0^{\pi} = 2\left(0 + \frac{2}{3} - 0 + \frac{2}{3}\right) =$

$= \frac{8}{3}.$

(Z) $\int_{-2}^{-1} \frac{dx}{x^2 + 4x + 5} = \int_{-2}^{-1} \frac{dx}{(x+2)^2 + 1} = \Big| \operatorname{arctg}(x+2) \Big|_{-2}^{-1} = \operatorname{arctg} 1 - \operatorname{arctg} 0 = \frac{\pi}{4}.$

(Z) $\int_{-2}^{-1} \frac{dx}{(11+5x)^3} \Rightarrow 11+5x=t,\ x=\frac{t-11}{5},\ dx=\frac{1}{5}dt, \Rightarrow$

$I=\frac{1}{5}\int\frac{dt}{t^3}=-\frac{1}{10}\cdot\frac{1}{t^2}=-\frac{1}{10}\frac{1}{(11+5x)^2}=-\frac{1}{10}\left|\frac{1}{(11+5x)^2}\right|_{-2}^{-1}=$

$=-\frac{1}{10}\left(\frac{1}{36}-1\right)=\frac{1}{10}\cdot\frac{35}{36}=\frac{7}{72}.$

(Z) $\int_{\ln 2}^{\ln 3}\frac{dx}{e^x-e^{-x}} \Rightarrow e^x=t,\ e^x dx=dt,\ dx=\frac{dt}{e^x}=\frac{dt}{t}.$

Za $x=\ln 2 \Rightarrow t=2$
$x=\ln 3 \Rightarrow t=3$

$\Rightarrow I=\int_2^3\frac{dt}{t(t-\frac{1}{t})}=\int_2^3\frac{dt}{t^2-1}=\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|_2^3=\frac{1}{2}\left(\ln\frac{2}{4}-\ln\frac{1}{3}\right)=$

$=\frac{1}{2}\ln\frac{\frac{1}{2}}{1/3}=\frac{1}{2}\ln\frac{3}{2}=\frac{1}{2}\ln 1{,}5.$

8. $\int_{-a}^{+a}f(x)dx=2\int_0^{+a}f(x)\,dx$, ako je $f(x)$ parna f-ja

$\int_{-a}^{+a}f(x)dx=0$, ako je $f(x)$ neparna f-ja.

9. $\int_a^b u\,dv=|uv|_a^b-\int_a^b v\,du=u(b)\cdot v(b)-u(a)\cdot v(a)-\int_a^b v\,du.$

(Z) $\int_0^{\pi/2a}(x+3)\sin ax\,dx=\left|u=x+3,\ du=dx,\ dv=\sin ax\,dx,\ v=-\frac{\cos ax}{a}\right|=$

$=-\frac{x+3}{a}\cos ax\Big|_0^{\pi/2a}+\frac{1}{a}\int_0^{\pi/2a}\cos ax\,dx=-\frac{1}{a}\left[\left(\frac{\pi}{2a}+3\right)\cdot 0-3\right]+\frac{1}{a^2}\left|\sin ax\right|_0^{\pi/2a}=$

$=\frac{3}{a}+\frac{1}{a^2}.$

10. Neka je $f(x)$ neprekidna na int. $[a,b]$ i neka je $m\le f(x)\le M$, tada u intervalu $[a,b]$ postoji tačka c takva da je:

$\int_a^b f(x)dx\le(b-a)f(c)$ gdje je $m\le f(c)\le M$, $f(c)$ je srednja vrijednost f-je $f(x)$ u intervalu $[a,b]$.

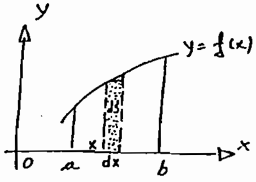
Odavde slijedi: $m(b-a)\le\int_a^b f(x)dx\le M(b-a)$

$\Rightarrow f(c)=\frac{1}{b-a}\int_a^b f(x)\,dx$.

$$\Rightarrow f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$ – srednja vrijednost f-je. $f(x)$.

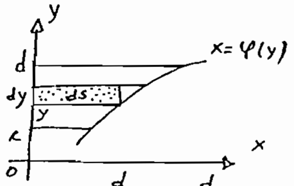
11. $$S=\int_a^b f(x)\,dx$$ – daje površinu omeđenu f-jom $f(x)$ i osom x.

Ako trebamo površinu omeđenu $f(x)$ i osom y prelazimo na inverznu f-ju od $f(x)$.



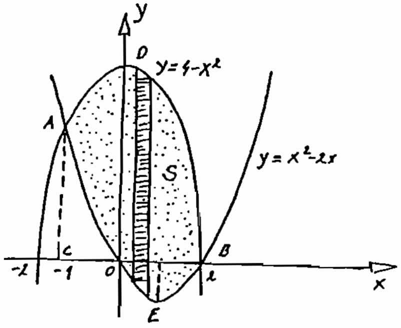
$ds = x\cdot dy$

$$S=\int_a^b y\,dx=\int_a^b f(x)\,dx.$$



$$S=\int_c^d x\,dy=\int_c^d \varphi(y)\,dy.$$

Primjer: Lik je omeđen parabolama $y=4-x^2$ i $y=x^2-2x$. Prikažemo li jednadžbe parabola u obliku $y=(2+x)(2-x)$ i $y=x(x-2)$ lako crtamo lik. Riješimo zadane parabole i nađemo sjecišta $A(-1;3)$ i $B(2;0)$. $S = CDOA + OEB - COA$.



$$S=\int_{-1}^{2}(4-x^2)\,dx+\int_{2}^{0}(x^2-2x)\,dx-\int_{-1}^{0}(x^2-2x)\,dx=$$

$$=\left|4x-\frac{x^3}{3}\right|_{-1}^{2}+\left|\frac{x^3}{3}-x^2\right|_{2}^{0}-\left|\frac{x^3}{3}-x^2\right|_{-1}^{0}=$$

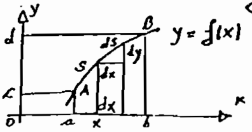
$$=9+\frac{4}{3}-\frac{4}{3}=9.$$

Ovaj zadatak smo mogli riješiti i ovako:

$$ds=(y_2-y_1)\,dx=(4-x^2-x^2+2x)\,dx=(-2x^2+2x+4)\,dx$$

$$S=\int_{-1}^{2}(-2x^2+2x+4)\,dx=\left|-\frac{2}{3}x^3+x^2+4x\right|_{-1}^{2}=-6+15=9.$$

12. Dužina luka krivulje (rektifikacija luka krivulje).



$ds^2 = dx^2 + dy^2$

$ds = \sqrt{dx^2+dy^2}$

$ds = \sqrt{1+y'^2_{(x)}}\,dx = \sqrt{1+x'^2_{(y)}}\,dy$

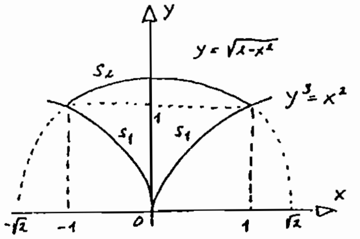
$$\overline{AB} = S = \int_a^b \sqrt{1+y'^2_{(x)}}\,dx$$

$$S=\int_c^d \sqrt{1+x'^2_{(y)}}\,dy$$

Dužina luka krivulje: $s=\int_a^b\sqrt{1+y'(x)^2}\,dx$.

Primjer: Odrediti obim lika koji je omeđen krivuljama $y^3=x^2$ i $y=\sqrt{2-x^2}$.

Rješavajući zajedno zadane jednadžbe, dobivamo sjecišta: $(1,1)$ i $(-1,1)$.



$S=2S_1+S_2$

$S_1=\int_0^1\sqrt{1+(\frac{3}{2}y^{1/2})^2}\,dy=\frac{1}{2}\int_0^1\sqrt{4+9y}\,dy=$

$=\frac{1}{27}\left|\sqrt{(4+9y)^3}\right|_0^1=\frac{1}{27}(13\sqrt{13}-8)$.

$S_2=\frac{1}{4}$ dužina kružnice poluprečnika $\sqrt{2}=\frac{1}{4}2\sqrt{2}\pi=\frac{\sqrt{2}}{2}\pi$.

$S=\frac{2}{27}(13\sqrt{13}-8)+\frac{\sqrt{2}}{2}\pi=5{,}102$.

12. Volumen:

Računa se jednostrukim integralom, ako površinu pop. presjeka možemo predstaviti kao f-ju njegova položaja, ili ako je tijelo rotaciono.

$dV=S(x)dx \Rightarrow V=\int_a^b S(x)dx \Rightarrow S(x)$ - površina pop. presjeka tijela zadana kao f-ja od x.

13. Jednadžbe rotacionih površina:

$y=f(x)$ oko ose x $\Rightarrow y^2+z^2=[f(x)]^2$

$x=\varphi(y)$ oko ose y $\Rightarrow x^2+z^2=[\varphi(y)]^2$ - uzimamo inv. f-ju od f-je $f(x)$.

14. Volumen (obujam) rotacionih tijela:

$S(x)=\pi y^2=\pi[f(x)]^2$

$V=\pi\int_a^b f(x)^2dx=\pi\int_a^b y(x)^2dx$

oko x ose.

rotacija krive $\varphi(y)=x$ oko y ose:

$V=\pi\int_c^d \varphi(y)^2dy=\pi\int_c^d x^2(y)\,dy$

15. Određivanje površina nastalih rotacijom krivih.

Ako krivulja $y=f(x)$ rotira oko ose X od $x=a$ do $x=b$, ona opisuje plohu, kojoj je površina $S=2\pi\int_a^b f(x)\sqrt{1+[f'(x)]^2}\,dx$.

Rotacija krive $x=\varphi(y)$ oko ose y od $y=c$ do $y=d$ površina je:

$$S=2\pi\int_c^d \varphi(y)\sqrt{1+[\varphi'(y)]^2}\,dy.$$

Primjer: Luk kubne parabole $y=x^3$ rotira oko ose x od $x=-\frac{2}{3}$ do $x=\frac{2}{3}$. Dobivamo:

$$S=2\cdot 2\pi\int_0^{2/3} x^3\sqrt{1+9x^4}\,dx$$ – jer uzimamo da se rotaciona ploha raspada na dva jednaka dijela.

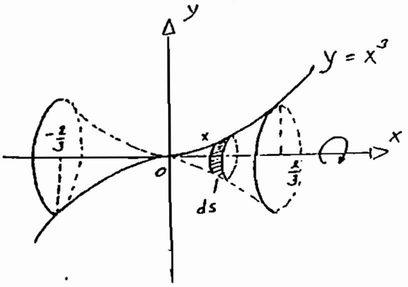
$$ds=\sqrt{1+[f'(x)]^2}=\sqrt{1+(3x^2)^2}$$

$1+9x^4=z$, $dz=36x^3dx$, $x^3dx=\frac{dz}{36}$,

za $x=0 \Rightarrow z=1$

za $x=\frac{2}{3} \Rightarrow z=\frac{25}{9}$

$$S=\frac{4}{36}\pi\int_1^{\frac{25}{9}}\sqrt{z}\,dz=\frac{\pi}{9}\left|\frac{2}{3}\sqrt{z^3}\right|_1^{\frac{25}{9}}=\frac{196}{729}\pi=0{,}846.$$



16. Nepravi integrali: (I slučaj)

1. $\int_a^{\infty} f(x)dx=\lim_{x\to\infty}\int_a^{x} f(x)dx$

2. $\int_{-\infty}^{b} f(x)dx=\lim_{x\to-\infty}\int_x^{b} f(x)dx$

3. $\int_{-\infty}^{+\infty} f(x)dx=\lim_{x\to-\infty}\int_x^{c} f(x)dx+\lim_{x\to+\infty}\int_c^{x} f(x)dx$

gdje je: $-\infty<c<+\infty$.

(II slučaj):

Podintegralna f-ja $f(x)$ neprekidna je za sve vrijednosti x u intervalu $[a;b]$ osim tačke c u kojoj $f(x)$ ima tačku prekida II vrste. Tada je:

$$\int_a^b f(x)dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\delta \to 0} \int_{c+\delta}^{b} f(x)dx.$$

ako ti limesi postoje i konačni su!

Ustvari imamo sljedeće slučajeve koji se pojavljuju u zadacima:

a) $f(x) \to \infty$, kad $x \to c$, $a < c < b$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0} \int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon_1 \to 0} \int_{c+\varepsilon_1}^{b} f(x)dx$$

b) $f(x) \to \infty$, kad $x \to a+0$ tj., teži donjoj granici integracije s desna.

$$\int_a^b f(x)dx = \lim_{x \to a+0} \int_x^b f(x)\,dx$$

c) $f(x) \to \infty$, kad $x \to b-0$ tj., teži gornjoj granici integracije slijeva:

$$\int_a^b f(x)dx = \lim_{x \to b-0} \int_a^x f(x)\,dx.$$

* Ako u tački $x=a$, bar jedna od jednostranih graničnih vrijednosti ne postoji, ili je ona ∞, tačka $x=a$ naziva se tačka prekida II vrste.

KARAKTERISTIČNI ODREĐENI INTEGRALI

(Z) $$I=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}\,dx=\left|\begin{array}{l}x=\pi-t,\\ dx=-dt\end{array}\right|\ \begin{array}{c|c|c} x & 0 & \pi\\ \hline t & \pi & 0\end{array}\Rightarrow -\int_{\pi}^{0}\frac{(\pi-t)\sin(\pi-t)\,dt}{1+\cos^2(\pi-t)}=$$

$$=\int_0^{\pi}\frac{(\pi-t)\sin t\,dt}{1+\cos^2 t}=\pi\int_0^{\pi}\frac{\sin t\,dt}{1+\cos^2 t}-\underbrace{\int_0^{\pi}\frac{t\sin t\,dt}{1+\cos^2 t}}_{I}\Rightarrow$$

$$\Rightarrow\ 2I=\pi\int_0^{\pi}\frac{\sin t\,dt}{1+\cos^2 t}\Rightarrow\ \begin{array}{l}\cos t=z\\ dz=-\sin t\,dt\Rightarrow dt=-\dfrac{dz}{\sin t}\end{array}$$

$$2I=\pi\int_0^{\pi}\frac{-\sin t\,dz}{\sin t\,(1+z^2)}$$

$$I=\frac{\pi}{2}\int_0^{\pi}-\frac{dz}{(1+z^2)}=-\frac{\pi}{2}\int_0^{\pi}\frac{dz}{(z^2+1)}=-\frac{\pi}{2}\left(\operatorname{arctg} z\right)\Big|_0^{\pi}=$$

$$=-\frac{\pi}{2}\left(\operatorname{arctg}(\cos t)\right)\Big|_0^{\pi}=-\frac{\pi}{2}\left(-\frac{\pi}{4}-\frac{\pi}{4}\right)=-\frac{\pi}{2}\left(-\frac{\pi}{2}\right)=\frac{\pi^2}{4}$$

$$I=\frac{\pi^2}{4}\ /$$

(Z) $$\int_0^{\pi}\frac{dx}{1+\alpha\cos x},\ (|\alpha|<1)$$

* $$1+\alpha\cos x=1+\alpha\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)=1+\alpha\cos^2\frac{x}{2}-\alpha\sin^2\frac{x}{2}=$$

$$=\underbrace{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}_{1}+\alpha\cos^2\frac{x}{2}-\alpha\sin^2\frac{x}{2}=\cos^2\frac{x}{2}(1+\alpha)+\sin^2\frac{x}{2}(1-\alpha)=$$

$$=\sin^2\frac{x}{2}(1-\alpha)+\cos^2\frac{x}{2}(1+\alpha).$$

$$\int_0^{\pi}\frac{dx}{1+\alpha\cos x}=\int_0^{\pi}\frac{dx}{(1-\alpha)\sin^2\frac{x}{2}+(1+\alpha)\cos^2\frac{x}{2}}\ \Big/:\cos^2\frac{x}{2}\ =$$

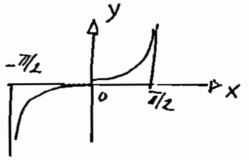
$$=\int_0^{\pi}\frac{\frac{dx}{\cos^2\frac{x}{2}}}{(1-\alpha)\operatorname{tg}^2\frac{x}{2}+(1+\alpha)}=\left|\operatorname{tg}\frac{x}{2}=t\Rightarrow dt=\frac{dx}{\cos^2\frac{x}{2}}\right|=$$

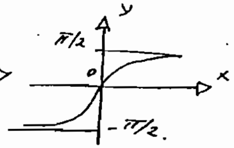
$$= \int_0^{\pi} \frac{dt}{(1-\alpha)t^2+(1+\alpha)} = \int_0^{\pi} \frac{dt}{(1-\alpha)\left(t^2+\frac{1+\alpha}{1-\alpha}\right)} = \frac{1}{1-\alpha}\int_0^{\pi} \frac{dt}{t^2+\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\right)} =$$

$$= \frac{1}{1-\alpha}\int_0^{\pi} \frac{dt}{t^2+\left(\sqrt{\frac{1+\alpha}{1-\alpha}}\right)^2} = \frac{1}{(1-\alpha)}\,\frac{2}{\sqrt{\frac{1+\alpha}{1-\alpha}}}\left(\operatorname{arctg}\frac{t}{\sqrt{\frac{1+\alpha}{1-\alpha}}}\right)\Bigg|_0^{\pi} =$$

$$= \frac{2}{1-\alpha}\cdot\frac{1}{\sqrt{\frac{1+\alpha}{1-\alpha}}}\left(\operatorname{arctg}\frac{\operatorname{tg}\frac{x}{2}}{\sqrt{\frac{1+\alpha}{1-\alpha}}}\right)\Bigg|_0^{\pi} = \frac{2}{(1-\alpha)\sqrt{\frac{1+\alpha}{1-\alpha}}}\left(\operatorname{arctg}\frac{\operatorname{tg}\frac{\pi}{2}}{\sqrt{\frac{1+\alpha}{1-\alpha}}} - 0\right) =$$

$$= \frac{2}{(1-\alpha)\sqrt{\frac{1+\alpha}{1-\alpha}}}\left(\operatorname{arctg}(\infty) - 0\right) = \frac{2}{(1-\alpha)\sqrt{\frac{1+\alpha}{1-\alpha}}}\cdot\frac{\pi}{2} = \frac{(\sqrt{1-\alpha})\cdot\pi}{(1-\alpha)\sqrt{1+\alpha}} =$$

* $\operatorname{tg} x \Rightarrow$ 

$\operatorname{arctg} x \Rightarrow$ 

$$= \frac{\sqrt{1-\alpha}\sqrt{1-\alpha}\cdot\pi}{(1-\alpha)\sqrt{1-\alpha}\sqrt{1+\alpha}} = \frac{(1-\alpha)\pi}{(1-\alpha)\sqrt{1-\alpha^2}} = \frac{\pi}{\sqrt{1-\alpha^2}}$$

(Z) $$\int_{-1}^{1} \frac{dx}{x^2-2x\cos\alpha+1} = \int_{-1}^{1} \frac{dx}{(x-\cos\alpha)^2+1-\cos^2\alpha} = \int_{-1}^{1} \frac{dx}{(x-\cos\alpha)^2+\sin^2\alpha} =$$

standardno
namještanje na puni kvadrat

$$= \frac{1}{\sin\alpha}\left(\operatorname{arctg}\frac{x-\cos\alpha}{\sin\alpha}\right)\Bigg|_{-1}^{1} = \frac{1}{\sin\alpha}\left(\operatorname{arctg}\frac{1-\cos\alpha}{\sin\alpha} - \operatorname{arctg}\frac{-1-\cos\alpha}{\sin\alpha}\right) =$$

$$= \frac{1}{\sin\alpha}\left(\operatorname{arctg}\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} - \operatorname{arctg}\frac{-(2\cos^2\frac{\alpha}{2})}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) =$$

$$= \frac{1}{\sin\alpha}\left(\operatorname{arctg}\frac{2\sin^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}} + \operatorname{arctg}\frac{2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}\right) =$$

$$= \frac{1}{\sin\alpha}\left(\operatorname{arctg}\left(\operatorname{tg}\frac{\alpha}{2}\right) + \operatorname{arctg}\left(\frac{1}{\operatorname{tg}\frac{\alpha}{2}}\right)\right) = \frac{1}{\sin\alpha}\left(\frac{\alpha}{2} + \operatorname{arctg}\left(\operatorname{ctg}\frac{\alpha}{2}\right)\right) =$$

$$= \left|\operatorname{arctg} x = \frac{\pi}{2} - \operatorname{arcctg} x\right| = \frac{1}{\sin\alpha}\left(\frac{\alpha}{2} + \frac{\pi}{2} - \operatorname{arcctg}\left(\operatorname{ctg}\frac{\alpha}{2}\right)\right) =$$

$$= \frac{1}{\sin\alpha}\left(\frac{\alpha}{2} + \frac{\pi}{2} - \frac{\alpha}{2}\right) = \frac{\pi}{2\sin\alpha}$$

ⓩ $\int_0^2 |1-x|\,dx = \int_0^1 (1-x)\,dx - \int_1^2 (1-x)\,dx = \int_0^1 dx - \int_0^1 x\,dx - \int_1^2 dx + \int_1^2 x\,dx =$

$= (1-0) - \left(\frac{1}{2} - 0\right) - (2-1) + \left(\frac{4}{2} - \frac{1}{2}\right) = 1 - \frac{1}{2} - 1 + \frac{3}{2} = \underline{1}$ /

ⓩ $\int_0^{\pi/4} \frac{sgn(ab)\,dx}{a^2\sin^2 x + b^2\cos^2 x} \;/ : \cos^2 x = \int_0^{\pi/4} \frac{\frac{sgn(ab)\,dx}{\cos^2 x}}{a^2 tg^2 x + b^2} =$

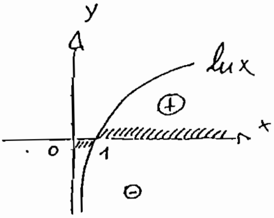
$= sgn(ab) \int_0^{\pi/4} \frac{\frac{dx}{\cos^2 x}}{a^2\left(tg^2 x + \left(\frac{b}{a}\right)^2\right)} = \left| \begin{matrix} tg\, x = t \\ dt = \frac{1}{\cos^2 x}\,dx \end{matrix} \right| =$

$= \frac{sgn(ab)}{a^2} \int_0^{\pi/4} \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2} = \frac{sgn(ab)}{a^2} \int_0^{\pi/4} \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2} =$

$= \frac{sgn(ab)}{a^2} \cdot \frac{1}{\left|\frac{b}{a}\right|} \left(arctg \frac{tg\, x}{\left|\frac{b}{a}\right|} \right) \Bigg|_0^{\pi/4} =$

$= \frac{sgn(ab)}{a \cdot b} \left(arctg \left|\frac{a}{b}\right| - 0 \right) = \underline{\frac{1}{a \cdot b}\, arctg \left|\frac{a}{b}\right|}$ /

ⓩ $\int_{1/e}^{e} |\ln x|\,dx = -\int_{1/e}^{1} \ln x\,dx + \int_1^e \ln x\,dx =$



$= -(x\ln x - x)\Big|_{1/e}^{1} + (x \ln x - x)\Big|_1^e =$

$= -\left(-1 + \frac{1}{e} + \frac{1}{e}\right) + (e - e + 1) = \underline{2\left(1 - \frac{1}{e}\right)}$ /

ⓩ $I_n = \int_0^1 (1-x^2)^n\,dx\,, \quad (n = 1, 2, \dots)$

$I_n = \int_0^1 (1-x^2)^n\,dx = \left| \begin{matrix} (1-x^2)^n = u\,, & du = n(1-x^2)^{n-1}(-2x)\,dx \\ dv = dx\,, & du = -2xn(1-x^2)^{n-1}\,dx \end{matrix} \right| =$

$= (1-x^2)^n \cdot x \Big|_0^1 - \int_0^1 x(-2x)\,n(1-x^2)^{n-1}\,dx =$

$$= 0 + 2n\int_0^1 x^2(1-x^2)^{n-1}dx = 2n\int_0^1 \left(x^2(1-x^2)^{n-1}\right)dx =$$

$$= 2n\int_0^1 (1-x^2)^{n-1}dx - 2n\int_0^1 (1-x^2)^n\,dx = 2n\,I_{n-1} - 2n\,I_n$$

$$* \quad x^2(1-x^2)^{n-1}dx = \left((1-x^2)^{n-1} - (1-x^2)^n\right)dx =$$

$$= (1-x^2)^{n-1}\left(1-1+x^2\right)dx = (1-x^2)^{n-1}\cdot x^2 dx$$

$$I_n = \frac{2n}{2n+1} I_{n-1} \quad (1,2,\dots)$$

$$I_n = \frac{2n}{2n+1}\cdot\frac{2(n-1)}{2(n-1)+1}\cdot I_{n-2}$$

$$I_n = \frac{2n}{2n+1}\cdot\frac{2(n-1)}{2(n-1)+1}\cdot\frac{2(n-2)}{2(n-2)+1} I_{n-3}$$

$$I_n = \frac{2n}{2n+1}\cdot\frac{2(n-1)}{2(n-1)+1}\cdot\frac{2(n-2)}{2(n-2)+1}\cdot\ldots\cdot\frac{2\cdot 2}{2\cdot 2+1} I_1 =$$

$$* \quad I_1 = \int_0^1 (1-x^2)dx = \left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{2}{3}$$

$$= \frac{2^n\cdot 2^n (n!)^2}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

(Z) $\int_{-a}^{a} x^2 \sin x\,dx$

$f(x) = x^2\sin x$

$f(-x) = -f(x)$ – f-ja je neparna

Stoga je: $\int_{-a}^{a} x^2\sin x\,dx = 0.$

(Z) $\int_{-1}^{1} \ln\frac{1+\sin x}{1-\sin x}\,dx$

$f(x) = \ln\frac{1+\sin x}{1-\sin x}$

$f(-x) = \ln\frac{1-\sin x}{1+\sin x} = -\ln\frac{1+\sin x}{1-\sin x} = -f(x).$

$$\Rightarrow \int_{-1}^{1} \ln\frac{1+\sin x}{1-\sin x} = 0.$$

(Z) $\int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx = \left| \begin{matrix} x = \frac{\pi}{2} - t, & x \,|\, 0 \,|\, \pi/2 \\ dt = -dx, & t \,|\, \pi/2 \,|\, 0 \end{matrix} \right| =$

$= -\int_{\pi/2}^{0} \frac{dt\,\cos^3(\frac{\pi}{2}-t)}{\cos^3(\frac{\pi}{2}-t)+\sin^3(\frac{\pi}{2}-t)} = \int_0^{\pi/2} \frac{\sin^3 t}{\cos^3 t + \sin^3 t}\,dt \Rightarrow$

$\int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\,dx = \int_0^{\pi/2} \frac{\sin^3 x}{\cos^3 x + \sin^3 x}\,dx \Rightarrow$

$2\int_0^{\pi/2} \frac{\cos^3 x\,dx}{\sin^3 x + \cos^3 x} = \int_0^{\pi/2} dt = \frac{\pi}{2}$

$\int_0^{\pi/2} \frac{\cos^3 x\,dx}{\cos^3 x + \sin^3 x} = \underline{\frac{\pi}{4}}$

(Z) $\int_0^{4\pi} \sqrt{1-\cos x}\,dx = \int_0^{4\pi} \sqrt{2\sin^2\frac{x}{2}}\,dx = \sqrt{2}\int_0^{4\pi} \left|\sin\frac{x}{2}\right| dx =$

$= \sqrt{2}\left(\int_0^{2\pi} \sin\frac{x}{2}\,dx - \int_{2\pi}^{4\pi} \sin\frac{x}{2}\,dx \right) = \sqrt{2}\left(-2\cos\frac{x}{2}\Big|_0^{2\pi} + 2\cos\frac{x}{2}\Big|_{2\pi}^{4\pi}\right) =$

$= \underline{8\cdot\sqrt{2}}$

(Z) $I_1 = \int_0^{\pi/2} \frac{\sin t}{\sin t + \cos t}\,dt = -\int_{\pi/2}^{0} \frac{\cos x\,dx}{\cos x + \sin x} = \int_0^{\pi/2} \frac{\cos x\,dx}{\cos x + \sin x} = I_2$

$\underbrace{I_1 = I_2}$ (за $\sin t$ — I_1; за $\cos t$ — I_2)

$2I_1 = I_1 + I_1 = I_1 + I_2 \Rightarrow$

$2I_1 = \int_0^{\pi/2} \frac{\sin t + \cos t}{\sin t + \cos t}\,dt = \int_0^{\pi/2} 1\cdot dt = \int_0^{\pi/2} dt = \frac{\pi}{2}$

$2I_1 = \frac{\pi}{2} \Rightarrow \underline{I_1 = \frac{\pi}{4}}$

(Z) $\int_1^2 \frac{dx}{1+\sqrt{2x-x^2}} = \int_1^2 \frac{dx}{1+\sqrt{1-(1-x)^2}} = \left| \begin{matrix} 1-x = \sin t \\ -dx = \cos t\, dt \end{matrix} \right| =$

* $(1-x)^2 = 1-2x+x^2$

$1-(1-x)^2 = 1-(1-2x+x^2) = 2x-x^2$

$= -\int_0^{-\frac{\pi}{2}} \frac{\cos t}{1+\cos t}\, dt = -\int_0^{-\pi/2} \frac{2\cos^2\frac{t}{2}-1}{2\cos^2\frac{t}{2}}\, dt =$

$= -\int_0^{-\pi/2} dt + \int_0^{-\pi/2} \frac{dt}{2\cos^2\frac{t}{2}} = -(-\pi/2 - 0) + \frac{1}{2}\int_0^{-\pi/2} \frac{dt}{\cos^2\frac{t}{2}} =$

$= \frac{\pi}{2} + \left(\operatorname{tg}\frac{x}{2}\right)\Big|_0^{-\pi/2} = \pi/2 + \left(\operatorname{tg}\left(-\frac{\pi}{4}\right) - 0\right) = \frac{\pi}{2}+1$

(Z) $I = I(t) = \int_0^{\pi} \frac{\sin x}{\sqrt{1-2t\cos x+t^2}}\, dx$

$t^2 - 2t\cos x + 1 = 0$

$\cos x = \frac{-1-t^2}{-2t}$

$\cos x = \frac{1+t^2}{2t} \Rightarrow \frac{2t\cos x}{(+)} = \frac{1+t^2}{(+)}$

$t_{1,2} = \frac{2\cos x \pm \sqrt{4\cos^2 x - 4}}{2} = \cos x \pm \sqrt{\cos^2 x - 1}$

$\cos^2 x - 1 \leq 0 \Rightarrow$ da možemo imati smjenu:

$1-2t\cos x + t^2 = u^2$

$dx = \frac{u\,du}{t\sin x}$

Za $x=0$ $\quad u^2 = (1-t)^2 \Rightarrow u = \sqrt{(1-t)^2} = |1-t|$

$u = |1-t|$

$u = \begin{cases} 1-t, & t \in (-\infty,0) \cup (0,1) \\ -(1-t), & t > 1 \end{cases}$

za $x=\pi \Rightarrow u^2=(1+t)^2 \Rightarrow u=\sqrt{(1+t)^2}=|1+t|$

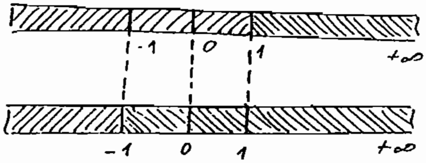
$u=|1+t|$

$$u=\begin{cases} 1+t, & t\in(-1,0)\cup(0,+\infty) \\ -(1+t), & t\in -1 \end{cases}$$

$$I=\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{1-2t\cos x+t^2}}=\left|\begin{matrix}\text{imamo}\\ \text{intervale}\end{matrix}\right|=\frac{1}{t}\int du \quad \left|\begin{matrix} t\in(-1,0)\cup(0,1)\\ t\in(-\infty,-1)\\ t\in(1,+\infty)\end{matrix}\right.$$

$x=0$

$x=\pi$



$$=\frac{1}{t}\int_{1-t}^{1+t}du=\frac{1}{t}(u)\Big|_{1-t}^{1+t}=2,\quad t\in(-1,1)$$

$$=\frac{1}{t}\int_{t-1}^{-1-t}du=\frac{1}{t}u\Big|_{1-t}^{-1-t}=-\frac{2}{t}\quad t\in(-\infty,-1)$$

$$=\frac{1}{t}\int_{t-1}^{1+t}du=\frac{2}{t},\quad t\in(1,+\infty)$$

$$I(1)=\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{2-2\cos x}}=\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{2}\sqrt{1-\cos x}}=\frac{1}{\sqrt{2}}\int_0^{\pi}\frac{\sin x\,dx}{\sqrt{2\sin^2\frac{x}{2}}}=$$

$$=\frac{1}{2}\int_0^{\pi}\frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}\,dx}{\sin\frac{x}{2}}\,dx=\int_0^{\pi}\cos\frac{x}{2}\,dx=2$$

Ⓩ $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\sqrt{\cos x}\,dx$ – f-ja je parna $\Rightarrow$

$I = 2\int_0^{\pi/2} |\sin x|\sqrt{\cos x}\,dx$ – gubimo modul jer je od 0 do $\frac{\pi}{2}$ $\sin x \geq 0$.

$= 2\int_0^{\pi/2} \sin x\sqrt{\cos x}\,dx \Rightarrow \left|\begin{matrix}\cos x = t\\ -\sin x\,dx = dt\end{matrix}\right| = 2\int_0^1 \sqrt{t}\,dt = \frac{4}{3}$

Ⓩ $I = \int_0^{\pi/2} x\left(\frac{\sin x}{1+\cos^2 x} + \frac{\cos x}{1+\sin^2 x}\right)dx = \left|\begin{matrix}x = \frac{\pi}{2} - t\\ dx = -dt\end{matrix}\right| =$

$= -\int_{\pi/2}^{0} \left(\frac{\pi}{2} - t\right)\left(\frac{\cos t}{1+\sin^2 t} + \frac{\sin t}{1+\cos^2 t}\right)dt =$

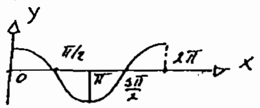
$= \frac{\pi}{2}\int_0^{\pi/2} \left(\frac{\cos x}{1+\sin^2 x} + \frac{\sin x}{1+\cos^2 x}\right)dx - I \Rightarrow$

$I = \frac{\pi}{4}\int_0^{\pi/2} \left(\frac{\cos x}{1+\sin^2 x} + \frac{\sin x}{1+\cos^2 x}\right)dx = \left|\begin{matrix}\sin x = t,\\ \cos x = t\end{matrix}\right| =$ itd...

ODREĐENI INTEGRALI PREKIDNIH FUNKCIJA

Ⓩ $\int_{-1}^{4} \frac{[x]}{1+|x|}dx = \int_{-1}^{0}\frac{-1}{1-x}dx + \int_0^1 \frac{0}{1+x}dx + \int_1^2 \frac{dx}{1+x} + \int_2^3 \frac{2dx}{1+x} + \int_3^4 \frac{3dx}{1+x} =$

$= 3\ln 5 - 2\ln 4 - \ln 3.$

Ⓩ $\int_0^{\pi} x\,\mathrm{sgn}(\cos x)\,dx = $ |  | =

$= \int_0^{\pi/2} x\,dy + \int_{\pi/2}^{\pi} -x\,dx = -\frac{\pi^2}{4}$

(Z) $\int_0^2 [e^x]\,dx = \int_0^{\ln 2} dx + \int_{\ln 2}^{\ln 3} 2\,dx + \dots + \int_{\ln 7}^{2} 7\,dx = 14 - \ln(7!)$.

dijelovi su $\ln 1, \ln 2, \ln 3, \dots$

$e^x = 1$ za $\underline{x=0}$

$1 \le e^x \le 2$ za $0 \le x \le \ln 2$

$\vdots$

$7 \le e^x \le 8$ za $\underline{x=2}$

* Što se tiče prekidne f-je u integralu, naglašavamo da se ona ne integrira, nego se njena vrijednost u određenom intervalu integrira.

(Z) $\int_{-1}^{1} \frac{dx}{1+x^2} = \int_{-1}^{0} \frac{dx}{1+x^2} + \int_0^1 \frac{dx}{1+x^2} = \left| \begin{matrix} x = \frac{1}{t} \\ dx = -\frac{1}{t^2}dt \end{matrix} \right| =$

$= -\int_{-1}^{-\infty} \frac{dt}{1+t^2} - \int_{+\infty}^{1} \frac{dt}{1+t^2} = \int_{-\infty}^{-1} \frac{dt}{1+t^2} + \int_1^{+\infty} \frac{dt}{1+t^2} =$

$= \lim_{a\to-\infty} \int_a^{-1} \frac{dt}{1+t^2} + \lim_{b\to+\infty} \int_1^b \frac{dt}{1+t^2} =$

$= \lim_{a\to-\infty} (\mathrm{arctg}\,)\Big|_a^{-1} + \lim_{b\to+\infty} (\mathrm{arctg}\, t)\Big|_1^b =$

$= -\frac{\pi}{4} - \left(-\frac{\pi}{2}\right) + \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} + \frac{\pi}{4} = \underline{\frac{\pi}{2}}$

(Z) $\int_{-2}^{2} \frac{dx}{\sqrt{|1-x^2|}} = \int_{-2}^{-1} \frac{dx}{\sqrt{x^2-1}} + \underbrace{\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}}}_{(1)} + \int_1^2 \frac{dx}{\sqrt{x^2-1}}$.

(1) $\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \lim_{x\to-1_+} \int_x^0 \frac{dx}{\sqrt{1-x^2}} + \lim_{x\to1_-} \int_0^x \frac{dx}{\sqrt{1-x^2}} =$

$= \lim_{x\to-1_+} (\arcsin 0 - \arcsin x) + \lim_{x\to1_-} (\arcsin x - \arcsin 0) = \frac{\pi}{2} + \frac{\pi}{2} = \underline{\pi}$

Funkcije više promjenjivih

F-je više promjenjivih

$z = f(x,y)$.

1° Pod područjem definicije f-je $z = f(x,y)$ podrazumijevamo skup tačaka (x,y) ravni xoy, u kojoj je dana f-ja definisana tj. prima realne vrijednosti.

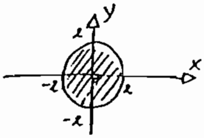
Primjer:

$$z = \frac{1}{\sqrt{4-x^2-y^2}} \Rightarrow \mathcal{D}(f): \; 4-x^2-y^2>0$$

$$-x^2-y^2>-4$$

$$\boxed{x^2+y^2<4}$$

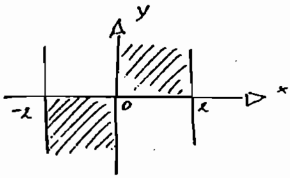
↳ kružnica u (0,0).



Primjer: $z = \arcsin\frac{x}{2} + \sqrt{xy}$

1° $\sqrt{xy} \Rightarrow xy \geq 0 \Rightarrow \begin{cases} x\geq 0 & x\leq 0 \\ y\geq 0, & y\leq 0 \end{cases}$

2° $-1 < \frac{x}{2} < 1 \Rightarrow -2<x<2$



2° Nivo linijom f-je $z=f(x,y)$ nazivamo takvu liniju $f(x,y)=c$ u ravnini xoy u čijim tačkama f-ja prima jednu te istu vrijednost $z=c$.

3° Nivo plohama f-je tri argumenta $u=f(x,y,z)$ nazivamo takvu plohu $f(x,y,z)=c$ u čijim tačkama f-ja prima konstantnu vrijednost $u=c$.

Primjer: Nacrtajmo nivo liniju f-je: $z=x^2y$

$x^2y = c \Rightarrow y = \frac{c}{x^2}$. Predpostavimo da je: $c=0,\pm1,\ldots$ i dobijemo porodicu nivo linija.

Neprekidnost

1° Limes f-je: $\lim_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$

Tipovi: $\lim_{\substack{x\to a\\ x\to +\infty}} f(x,y)$, $\lim_{\substack{x\to a\\ x\to -\infty}} f(x,y)$, $\lim_{\substack{x\to +\infty\\ y\to b}} f(x,y)$, $\lim_{\substack{x\to -\infty\\ y\to b}} f(x,y)$

$\lim_{\substack{x\to +\infty\\ y\to +\infty}} f(x,y)$, $\lim_{\substack{x\to +\infty\\ y\to -\infty}} f(x,y)$, $\lim_{\substack{x\to -\infty\\ x\to +\infty}} f(x,y)$, $\lim_{\substack{x\to -\infty\\ y\to -\infty}} f(x,y)$.

Sukcesivne granične vrijednosti:

$L_1 = \lim_{x\to a}\left(\lim_{y\to b} f(x,y)\right)$, $L_2 = \lim_{x\to b}\left(\lim_{x\to a} f(x,y)\right)$

$L = \lim_{\rho\to 0} f(x,y) = \lim_{\substack{x\to a\\ y\to b}} f(x,y)$ } dvojni limes.

$\rho = \rho(A,X) = \sqrt{(x-a)^2+(y-b)^2}$, $X=(x,y)$, $A=(a,b)$

$L = \lim_{\substack{x\to a\\ y\to b}} f(x,y)$ ne postoji ako:

1) $\exists\, L_1 i L_2$ a $L_1 \neq L_2$
2) L zavisi od ugla φ ako se pređe na polarne koordinate: $x = \rho\cos\varphi$
$y = \rho\sin\varphi$.

Primjer: Dokazati da postoje uzastopne granične vrijednosti i da su jednake, a da ipak ne postoji dvojna granična vrijednost:
$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y)$ ako je $f(x,y) = \frac{xy}{x^2+y^2}$

Dokaz:

$\lim_{y\to 0}\left[\lim_{\substack{x\to 0\\ y\neq 0}} f(x,y)\right] = \lim_{y\to 0}\left[\lim_{\substack{x\to 0\\ y\neq 0}} \frac{xy}{x^2+y^2}\right] = \lim_{y\to 0}\frac{0\cdot y}{0+y^2} = \underline{0}$.

$\lim_{x\to 0}\left[\lim_{\substack{y\to 0\\ x\neq 0}} f(x,y)\right] = \lim_{x\to 0}\left[\lim_{\substack{y\to 0\\ x\neq 0}} \frac{x\cdot 0}{x^2+0}\right] = \lim_{x\to 0}\frac{0\cdot 0}{0} = \underline{0}$.

ali je: $\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\rho\to 0}\frac{\rho^2\sin\varphi\cos\varphi}{\rho^2(\sin^2\varphi+\cos^2\varphi)} = \lim_{\rho\to 0}(\sin\varphi\cos\varphi) = \underline{\sin\varphi\cos\varphi}$

kako limes zavisi od ugla φ to limes ne postoji!

Primjer: Naći granične vrijednosti:

a) $\lim\limits_{\substack{x\to 1\\ y\to 1}} \frac{xy-x-y+1}{\sqrt{xy-x-y+2}-1}$ b) $\lim\limits_{\substack{x\to 0\\ y\to a}} \frac{\sin xy}{x}$

c) $\lim\limits_{\substack{x\to \infty\\ y\to a}} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}}$

Rj:

a) $\lim\limits_{\substack{x\to 1\\ y\to 1}} \frac{x(y-1)-(y-1)}{\sqrt{x(y-1)-(y-2)}-1} = \lim\limits_{\substack{x\to 1\\ y\to 1}} \frac{(y-1)(x-1)}{\sqrt{x(y-1)-(y-2)}-1} \Big/ \cdot \frac{\sqrt{x(y-1)-(y-2)}+1}{\sqrt{x(y-1)-(y-2)}+1}$

$= \lim\limits_{\substack{x\to 1\\ y\to 1}} \frac{(y-1)(x-1)\left(\sqrt{x(y-1)-(y-2)}+1\right)}{x(y-1)-(y-2)-1} = \lim\limits_{\substack{x\to 1\\ y\to 1}} \frac{(y-1)(x-1)\left(\sqrt{x(y-1)-(y-2)}+1\right)}{x(y-1)-(y-1)} =$

$= \lim\limits_{\substack{x\to 1\\ y\to 1}} \left(\frac{(y-1)(x-1)\left(\sqrt{x(y-1)-(y-2)}+1\right)}{(x-1)(y-1)}\right) = \lim\limits_{\substack{x\to 1\\ y\to 1}} \left(\sqrt{x(y-1)-(y-2)}+1\right) =$

$= \lim\limits_{\substack{x\to 1\\ y\to 1}} \left(\sqrt{1(y-1)-(y-2)}+1\right) = \sqrt{1}+1 = \underline{2}$!

b) $\lim\limits_{\substack{x\to 0\\ y\to a}} \frac{\sin xy}{x} = \lim\limits_{\substack{x\to 0\\ y\to a}} \frac{\sin xy}{x\cdot y}\cdot y = \lim\limits_{y\to a} 1\cdot y = a$

$xy \to 0$ za $x\to 0$, $y \to$ bilo kojem broju.

c) $\lim\limits_{\substack{x\to \infty\\ y\to a}} \left(1+\frac{1}{x}\right)^{\frac{x^2}{x+y}} = \lim\limits_{\substack{x\to +\infty\\ y\to a}} \left(\left(1+\frac{1}{x}\right)^{x}\right)^{\frac{x}{x+y}\}\to 1} = e^1 = \underline{e}$!

2° Neprekidnost: F-ju $z=f(x,y)$ nazivamo neprekidnom u tački $P(a,b)$, ako je $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b)$.

* F-ja $f: S\to R$ definirana na proizvoljnom skupu (neprazanom) $S\subseteq R^n$, neprekidna je u točki $A\in S$, ako $\forall\ \varepsilon>0\ \exists\ \delta=\delta(\varepsilon)>0$ takav da važi (2) $x\in S \wedge \rho(A,x)<\delta \Rightarrow |f(x)-f(A)|<\varepsilon$.

F-ja može biti prekidna u pojedinim točkama, i u točkama koje tvore jednu ili više linija prekidnosti, a prekod i složenije geometrijske linije (likove).

Primjer: Naći tačku prekidnosti f-je: $z = \frac{xy+1}{x^2-y}$!

$x^2-y \neq 0 \Rightarrow$

$x^2-y=0$ je tačka prekidnosti $\Rightarrow$ $\boxed{y=x^2}$ - ovo je skup tačaka parabole.

Za neprekidne f-je u oblasti definisanosti vrijede osobine:

1) Ako je D zatvorena ograničena oblast, onda je f ograničena u toj oblasti i postoji bar jedna tačka u kojoj f-ja ima max. i min. vrijednosti.

2) Ako je f neprekidna u zatvorenoj ograničenoj oblasti tada je ona uniformno neprekidna u toj oblasti.

Primjer: Ispitati neprekidnost f-je f date sa:

a) $f(x,y) = \arcsin(x+y)$

RJ:

$D(f)$: $|x+y| \leq 1$ - ovo je zatvorena ograničena oblast

$\Updownarrow$

$-1 \leq x+y \leq 1$

kako je $\lim\limits_{\substack{x\to a\\ y\to b}} f(x,y) = \arcsin(a+b) = f(a,b)$ $\forall (a,b) \in D$ pa je f-ja neprekidna u D.

b) $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$

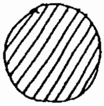
$\lim\limits_{\substack{x\to a\\ y\to b}} \left(\frac{xy^2}{x^2+y^2}\right) = \frac{ab^2}{a^2+b^2} = f(a,b)$ pa je u $R \setminus \{(0,0)\}$

f-ja neprekidna.

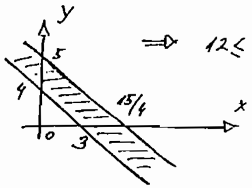
$\lim\limits_{\substack{x\to 0\\ y\to 0}} \left(\frac{xy^2}{x^2+y^2}\right) = 0 \neq 1$ pa je f-ja prekidna u tački $\underline{(0,0)}$.

* Otvorena oblast je sa znakom manje „<" tj. sve su tačke unutrašnje:

* Zatvorena oblast je oblast koja sadrži sve svoje tačke nagomilavanja ($\leq$).



Oba kruga predstavljaju OGRANIČENE skupove.
Neograničeni skup je ovakav:



$\Rightarrow 12 \leq 4x+3y \leq 15$

Primjer: Ispitati ravnomjernu neprekidnost f-je:

a) $f(x,y) = \sqrt{-x^2+2x-y^2-2y} + \operatorname{arccosec}(x^2-4x+y^2-6)$;

RJ: Definisana je na: $\begin{cases} -x^2+2x-y^2-2y \geq 0 \\ |x^2-4x+y^2-6| \geq 1 \end{cases} \Rightarrow$

$\Rightarrow (x-1)^2+(y+1)^2 \leq 2$

Ovo nam govori da je zatvoren skup D i da je ograničen. Ispitajmo neprekidnost.

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = \sqrt{-a^2+2a-b^2-2b} + \operatorname{arccosec}(a^2-4a+b^2-6) =$$

$= f(a,b)$. F-ja je neprekidna pa je uz gornji uslov zatvorenosti i uniformno neprekidna!

Primjer: Ispitati neprekidnost f-ja:

a) $(x,y) \to z = \dfrac{xy}{x^2+y^2}$

b) $(x,y) \to z = \dfrac{xy}{\sqrt{x^2+y^2}}$

RJ:

a) $z = \dfrac{xy}{x^2+y^2}$ $\qquad D(f): \forall x,y \in R \setminus \{(0,0)\}$.

Ova f-ja je neprekidna jer su joj faktori koji je grade elementarno neprekidni $D(f(x,y))$.
Međutim u tački $(0,0)$ f-ja je prekidna, ali ako tražimo limes za svaku promjenjivu posebno dobićemo:

$$\lim_{\substack{x\to 0\\ y\text{-fiksirano}}} \frac{0\cdot y}{0+y^2} = 0 \;,\; \lim_{\substack{y\to 0\\ x\text{-fiksirano}}} \frac{x\cdot 0}{x^2+0} = 0$$ Tj. našli smo:

$\lim_{x\to 0} f(x,0) = 0$ i $\lim_{y\to 0} (0,y) = 0$ pa ako definišemo $f(0,0) = 0$ dobićemo neprekidnu f-ju.
Tj., zaključujemo da je f-ja bila prekidna sa otklonjivim prekidom.

b) $z = \dfrac{xy}{\sqrt{x^2+y^2}}$ $\qquad D(f): \forall x,y \in R \setminus \{(0,0)\}$.

Neprekidna je, jer su joj faktori elementarno nep., osim u tački $(0,0)$.

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{y\to 0} f(0,y) = \lim_{x\to 0} f(x,0) =$$

$= \Big|$ prelazom na na $(x = \rho\cos\varphi,\ y = \rho\sin\varphi)\Big| =$

$= \lim_{\rho\to 0} \rho \cos\varphi \sin\varphi = 0$ pa f-ju možemo učiniti neprekidnom

na (0,0), ako stavimo $f(0,0)=0$

* Duž se uvijek nalazi unutar prostora - prostor je konveksan
* Duž izlazi van - konkavan

Parcijalni izvodi

Ako je $z = f(x,y)$ onda uz pretpostavku da je y konst. imamo parcijalnu derivaciju po x definisanu na slijedeći način:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = f'_{(x)}(x,y)$$

Analogno se definiše izvod f-je po promjenljivoj y.

Izvod f-je $f(x)$ - jedne realne promjenjive je:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}.$$

Pogledajmo parcijalni izvod f-je dvije realne promjenjive. On je potpuno isti samo još pišemo i y - ali kao konstantu:

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

Parcijalni izvod u tački $A(a,b)$ je:

$$\lim_{x \to a} \frac{f(x,y) - f(a,y)}{x-a}, \ (y=b),$$

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ili $\lim \frac{f(a+\Delta x, b) - f(a,b)}{\ldots}$

Primjer: Naći parcijalne izvode f-je:

a) $z = \sqrt[3]{x^3+y^3}$

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{(x+\Delta x)^3+y^3} - \sqrt[3]{x^3+y^3}}{\Delta x} = \text{itd.}..$$

Ovako bi išli preko definicije izvoda. Međutim, potrebno je poznavati teoriju rješavanja limesa koji je puno teži od poznavanja pravila za izvode. Prema pravilima imamo:

$$\frac{\partial z}{\partial x} = \left((x^3-y^3)^{\frac{1}{3}}\right)' = \frac{1}{3}(x^3-y^3)^{\frac{1}{3}-\frac{3}{3}} \cdot 3x^2 = \frac{x^2}{\sqrt[3]{(x^3-y^3)^2}}$$

$$\frac{\partial z}{\partial y} = \frac{y^2}{\sqrt[3]{(x^3-y^3)^2}}$$

Oba izraza nisu definisana u tački $(0,0)$. Zbog toga tražimo izvode po definiciji u tački $(0,0)$.

$$\frac{\partial z(0,0)}{\partial x} = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{(\Delta x)^3+0^3} - 0}{\Delta x} = \underline{\underline{1}}$$

$$\frac{\partial z(0,0)}{\partial y} = \lim_{\Delta y \to 0} \frac{f(0, 0+\Delta y) - f(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\sqrt[3]{0^3+(\Delta y)^3} - 0}{\Delta y} = \underline{\underline{1.}}$$

Primjer: Nađimo parcijalne derivacije f-je:

$z = \ln tg \frac{x}{y}$.

Rj:

$$\frac{\partial z}{\partial x} = \frac{1}{tg\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \cdot \frac{1}{y} = \frac{2}{y \sin\frac{2x}{y}}$$

$$\frac{\partial z}{\partial y} = \frac{1}{tg\frac{x}{y}} \cdot \frac{1}{\cos^2\frac{x}{y}} \left(-\frac{x}{y^2}\right) = -\frac{2x}{y^2 \sin\frac{2x}{y}}$$

Primjer: Naći parcijalne derivacije f-je tri argumenta:

$$u = x^3y^2z + 2x - 3y + z + 5$$

$$\frac{du}{dx} = 3x^2y^2z + 2$$

$$\frac{du}{dy} = 2x^3yz - 3$$

$$\frac{du}{dz} = x^3y^2 + 1$$

Diferencijabilnost funkcije

* Ako je f-ja diferencijabilna u tački A, tada ona u toj tački ima konačne izvode po svim argumentima i važi:

$$df(A,x) = f'_{(x_1)}(A)(x_1 - a_1) + \dots + f'_{x_n}(A)(x_n - a_n).$$

* Ako f-ja f(x) ima konačne i određene parcijalne izvode po svim argumentima u okolini tačke A i ako su ti izvodi neprekidne f-je u A, f(x) je diferencijabilna u A.

Totalni diferencijal f-je

* Totalnim priraštajem f-je $z = f(x,y)$ nazivamo razliku:
$\Delta z = \Delta f(x,y) = f(x+\Delta x, y+\Delta y) - f(x,y)$.
* Totalnim diferencijalom f-je $z = f(x,y)$ u tački (x,y) nazivamo glavni dio totalnog priraštaja Δz za $\Delta x \to 0$, $\Delta y \to 0$, koji je linearan s obzirom na priraštaje argumenta Δx i Δy.
* F-ja ima totalni diferencijal ako su njeni parcijalni izvodi neprekidni

* F-ja je diferencijabilna ako ima totalni diferencijal.

Totalni diferencijal f-je $z = f(x,y)$:

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

Primjer: Nadimo totalni diferencijal f-je:

$$z = f(x,y) = \sqrt{x^2+y^2}$$

$$\left.\begin{aligned} \frac{\partial z}{\partial x} &= \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}} \\ \frac{\partial z}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}\right\} \Rightarrow$$

$$dz = \frac{x\,dx}{\sqrt{x^2+y^2}} + \frac{y\,dy}{\sqrt{x^2+y^2}}$$

Deriviranje složenih f-ja

$z = f(x,y)$ za $x = \varphi(t)$, $\psi(t) = y$ $\Rightarrow$

$$z = f(\varphi(t), \psi(t))$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

Ako se t podudara sa jednim argumentom (recimo sa x) onda će biti:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

Primjer: Izračunati $\frac{dz}{dt}$ ako je $z = e^{3x+2y}$ gdje je:

$x = \cos t$
$y = t^2$

RJ:

$$\left.\begin{array}{ll} \frac{dx}{dt} = -\sin x, & \frac{\partial z}{\partial x} = e^{3x+2y} \cdot 3 \\ \frac{dy}{dt} = 2t, & \frac{\partial z}{\partial y} = e^{3x+2y} \cdot 2 \end{array}\right\} \Rightarrow$$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dx} =$$

$$= -3 \cdot e^{3x+2y} \cdot \sin t + 2e^{3x+2y} \cdot 2t$$

Za sve razmotrene slučajeve vrijedi formula:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Izvodi i diferencijali višeg reda

$$\frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x,y) \quad \text{itd,...}$$

Primjer: Naći parcijalne derivacije drugog reda:

$$z = \operatorname{arctg} \frac{x}{y}.$$

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y^2}{(y^2+x^2)y} = \frac{y}{y^2+x^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{-x}{y^2} = \frac{-x}{y^2+x^2}$$

$$\frac{\partial^2 z}{\partial x^2} = y\left(\frac{-2x}{(y^2+x^2)^2}\right) = \frac{-2xy}{(y^2+x^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = -x\left(\frac{-2y}{(y^2+x^2)^2}\right) = \frac{2xy}{(y^2+x^2)^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \left(\frac{y}{y^2+x^2}\right)_y = \frac{(y^2+x^2)-2y^2}{(y^2+x^2)^2} = \frac{y^2-2y^2+x^2}{(y^2+x^2)^2} =$$

$$= \frac{-y^2+x^2}{(y^2+x^2)^2}.$$

Diferencijal drugog reda f-je: $z = f(x,y)$.

$d^2 z = d(d\ z)$

$d^3 z = d(d^2 z)$

$$d^2 z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x \partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2$$

$d^n z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n z$ – simbolička formula koja se formalno razvija po binomnom zakonu.

(Z) Ispitati neprekidnost f-je:

$$f(x,y) = \begin{cases} x^2 y (x^4+y^2)^{-1}, & (x,y) \neq 0 \\ 0, & (x,y) = (0,0) \end{cases}$$

Rj:

F-ja ima prekid u $(x,y) = (0,0)$ ali on je uamješten.

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y}{x^4 + y^2} = ?$$

F-ja je elementarno neprekidna pa je neprekidna tamo gdje je definisana!

Potrebno je ispitati tačku $(0,0)$.

Tačka (0,0).

Neka je $x = \frac{1}{n}$, $y = \frac{1}{n}$, $n \to \infty$ ⇒ da tada $x \to 0$, $y \to 0$

pa je:
$$\lim_{n\to\infty} f\left(\frac{1}{n}, \frac{1}{n}\right) = \lim_{n\to\infty} \frac{n}{1+n^2} = 0$$

Neka je $x = \frac{1}{n}$, $y = \frac{1}{n^2}$, $n \to \infty$ opet je $x \to 0$, $y \to 0$ ali je

sada: $$\lim_{n\to+\infty} f\left(\frac{1}{n}, \frac{1}{n^2}\right) = \frac{1}{2}.$$

Granična vrijednost u tački nula ne postoji!

(2) Ispitati diferencijabilnost f-je:

$z = \sqrt[3]{x^3+y^3}$, a posebno tačku (0,0).

RJ:
$$\frac{\partial z}{\partial x} = \frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}, \quad \frac{\partial z}{\partial y} = \frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}, \quad (x,y) \neq (0,0).$$

Izvode tražimo po definiciji:

$$\frac{\partial z}{\partial x}(0,0) = \lim_{h\to 0} \frac{z(h,0) - z(0,0)}{h} = 1$$

$$\frac{\partial z}{\partial y}(0,0) = 1$$

$\frac{\partial z}{\partial x}$ i $\frac{\partial z}{\partial y}$ nisu definisani u svim tačkama $(-a, a)$

$$\left.\begin{aligned} \frac{\partial z}{\partial x}(-a,a) &= \lim_{h\to 0} \frac{\sqrt[3]{h^3+3h^2a+3ha^2}}{h} = +\infty \\ \frac{\partial z}{\partial y}(-a,a) &= \infty \end{aligned}\right\}$$

ne postoje parcijalni izvodi!

(Z) Ispitati neprekidnost:

$$z = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & (x,y)\neq(0,0). \\ 0, & (x,y)=0 \end{cases}$$

RJ:

$$\lim_{\substack{x\to a\\ y\to b}} f(x,y) = (a^2+b^2)\sin\frac{1}{a^2+b^2} = f(a,b)$$

za $(a,b)\neq(0,0)$. je neprekidna!

Neprekidnost u točki $(0,0)$.

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} (x^2+y^2)\sin\frac{1}{x^2+y^2} = f(0,0)$$ pa je

f-ja neprekidna.

(Z) Dokazati da je f-ja:

$$f(x,y) = \begin{cases} \frac{x^2y}{2(x^4+y^2)}, & \text{za } x\neq 0 \text{ i } y\neq 0 \\ 0, & \text{za } x=y=0. \end{cases}$$

neprekidna u točki $(0,0)$ duž svake poluprave $x=t\cos\varphi$, $y=t\sin\varphi$, $(0\leq t<+\infty)$ koja prolazi kroz točku $(0,0)$, a ipak je prekidna u točki $(0,0)$.

Dokaz:

$$\lim_{t\to 0} f(t\cos\varphi, t\sin\varphi) = \lim_{t\to 0}\frac{t^3\cos^2\varphi\sin\varphi}{2t^2(t^2\cos^4\varphi+\sin^2\varphi)} = 0 = f(0,0).$$

Dovoljno je da uzmemo $y=kx^2$, pa da dokažemo drugi dio tvrdnje zadatka jer je:

$$L = \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = \lim_{\substack{x\to 0\\ (y=kx)}} \frac{kx^4}{2(x^4+k^2x^4)} = \frac{k}{2(1+k^2)}$$

L zavisi od k tj., za $k=0$, $L=0$,
za $k=1$, $L=\frac{1}{4}$.

što znači da vrijednost od L zavisi kojom parabolom se približavamo.

② Ispitati neprekidnost, egzistenciju parcijalnih izvoda i diferencijabilnost f-je : $f(x,y)=\sqrt[3]{x^3+y^3}$.

RJ:

a) kako je $\lim\limits_{\substack{x\to a\\ y\to b}} = \sqrt[3]{a^3+b^3} = f(a,b)$, zaključujemo da je data fja neprekidna na cijeloj ravni xOy.

b) $f_x' = \dfrac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$, $f_y' = \dfrac{y^2}{\sqrt[3]{(x^3+y^3)^2}}$ za $(x,y)\neq(0,0)$.

$\lim\limits_{\substack{x\to 0\\ y\to 0}} f_x'$ i $\lim\limits_{\substack{x\to 0\\ y\to 0}} f_y'$ nepostoje pa su prekidni izvodi f_x' i f_y' u tački $(0,0)$.

f_x' i f_y' su neograničeni u okolini tačke $(0,0)$.

c) kako su parcijalni izvodi f_x' i f_y' neprekidni u svakoj tački $(x,y)\neq(0,0)$ to je f diferencijabilna u svakoj tački $(x,y)\neq(0,0)$. Ako bi f bila diferencijabilna i u tački $(0,0)$ onda bi moralo vrijediti:

$$f(x,y)-f(0,0) = f_x'(0,0)(x-0)+f_y'(0,0)(y-0)+\omega(x,y)\cdot\sqrt{(x-0)^2+(y-0)^2}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \omega(x,y) = \omega(0,0)=0.$$

kako je:

$$f_x'(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{\sqrt[3]{h^3}}{h} = \underline{1}$$

$f'_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{k}{k} = 1, \quad f(0,0) = 0.$

to $f(x,y) = 1\cdot x + 1\cdot y + \omega(x,y)\sqrt{x^2+y^2} \Rightarrow \omega(x,y) = \frac{f(x,y)-x-y}{\sqrt{x^2+y^2}} = \frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}}$

Vidimo da je

$\lim_{\substack{x\to 0\\ y\to 0}} \omega(x,y) \neq \omega(0,0) = 0$ jer $\lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}}$ ne teži nuli;

što se vidi ako uzmemo $y = kx$:

$$L(k) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{\sqrt[3]{x^3+y^3}-x-y}{\sqrt{x^2+y^2}} = \lim_{\substack{x\to 0\\ (y=kx)}} \frac{\sqrt[3]{1+k^3}-1-k}{\sqrt{1+k^2}} = \begin{cases} \vdots \\ 0 & \text{za } k=0 \\ \frac{\sqrt[3]{2}-2}{\sqrt{2}} & \text{za } k=1 \\ \vdots \end{cases}$$

Ovaj limes zavisi od k. Prema tome f-ja nije diferencijabilna u tački $(0,0)$.

(2) Ispitati da li je f-ja:

$$f(x,y) = \begin{cases} e^{\frac{-1}{x^2+y^2}}, & x^2+y^2 \neq 0 \\ 0, & x=y=0 \end{cases}$$

diferencijabilna u tački $(0,0)$.

RJ:

$$f'_x(0,0) = \frac{\partial f(0,0)}{\partial x} = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} =$$

$$= \lim_{h\to 0} \frac{e^{-\frac{1}{h^2}}}{h} = \lim_{h\to 0} \frac{\frac{1}{h}}{e^{1/h^2}} = \lim_{k\to\infty} \frac{k}{e^{k^2}} = \lim_{k\to\infty} \frac{1}{2ke^{k^2}} = 0$$

$f'_y(0,0) =$ na isti način je $= 0$.

$$\underbrace{f(x,y) - f(0,0)}_{0} = f'_x(0,0)(x-0) + f'_y(0,0)(y-0) + \omega(x,y)\cdot\sqrt{(x-0)^2+(y-0)^2}.$$

$$f(x,y) = \omega(x,y)\cdot\sqrt{x^2+y^2} \Rightarrow \omega(x,y) = \frac{f(x,y)}{\sqrt{x^2+y^2}} = \frac{e^{\frac{-1}{x^2+y^2}}}{\sqrt{x^2+y^2}}$$

Ako je f-ja neprekidna u izvodima, tj. ako su parcijalni izvodi neprekidne f-je i ako $\omega(x,y)\to 0$, kod $x\to 0$ i $y\to 0$ f-ja je u $(0,0)$ diferencijabilna!

$$\lim_{\substack{x\to 0\\ y\to 0}} \omega(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{e^{-\frac{1}{x^2+y^2}}}{\sqrt{x^2+y^2}} = \lim_{r\to 0_+} \frac{e^{-\frac{1}{r^2}}}{r} = \lim_{r\to 0+} \frac{1/r}{e^{(\frac{1}{r})^2}} =$$

$= \left| \frac{1}{r} = t,\ (r\to 0+) \Leftrightarrow t\to+\infty \right| = \lim_{t\to+\infty} \frac{t}{e^{t^2}} = \lim_{t\to+\infty} \frac{1}{e^{t^2}\cdot 2t} = 0,$

Ovo znači da je f-ja diferencijabilna u tački (0,0).

Napomena: Primijetimo da f-ja $f(x,y)$ ima konačne i određene parcijalne izvode u okolini (0,0), i da su oni neprekidni u (0,0). (Dovoljno je da jedan od par. izvoda postoji u samoj tački, a drugi u okolini tačke i da je neprekidan u toj tački).

$$\frac{\partial f(x,y)}{\partial x} = \frac{2x}{(x^2+y^2)^2} e^{-\frac{1}{x^2+y^2}}, \quad \frac{\partial f}{\partial y} = \frac{2y}{(x^2+y^2)^2} e^{-\frac{1}{x^2+y^2}}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial f(x,y)}{\partial x} = \frac{\partial f(0,0)}{\partial x} = 0$$

$$\lim_{\substack{x\to 0\\ y\to 0}} \frac{\partial f(x,y)}{\partial y} = \frac{\partial f(0,0)}{\partial y} = 0.$$

(Z) Dokazati da f-ja $f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^2}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$ nije diferencijabilna u (0,0).

Rj: Provjerimo da li su parcijalni izvodi (jedan od njih) neprekidni u tački (0,0). Pošto granične vrijednosti $\lim_{\substack{x\to 0\\ y\to 0}} f_x'(x,y)$, $\lim_{\substack{x\to 0\\ y\to 0}} f_y'(x,y)$ ne postoje jer je:

$$\left.\begin{array}{l} f_x'(x,y) = \frac{y^2(y^2-x^2)}{(x^2+y^2)^2} = \sin^2\varphi\cos^2\varphi \\ f_y'(x,y) = \ldots = 2\cos^3\varphi\sin\varphi \end{array}\right\}$$

to f_x' i f_y' nisu neprekidni u tački (0,0).

Međutim, ako su par. izvodi prekidni u nekoj tački to još ne znači da f-ja nije diferencijabilna u toj tački, jer neprekidnost parc. izvoda predstavlja samo dovoljan uslov diferencijabilnosti.

Provjerimo da li su ispunjeni potrebni uslovi diferencijabilnosti:

1° postoje konačni parcijalni izvodi:

$f'_x(0,0)$ i $f'_y(0,0)$

2° $df(X,A) = f'_x(0,0)(x-0) + f'_y(0,0)(y-0)$.

Prvi uslov je ispunjen, jer je:

$$f'_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

$$f'_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

Kada bi f-ja bila diferencijabilna u (0,0), njen totalni diferencijal bi morao da bude jednak nuli u ovoj tački i još bi moralo da bude:

$$\lim_{X\to(0,0)} \omega(X) = \lim_{r\to 0} \frac{\Delta f(X,A) - df(X,A)}{r} = 0 \quad , \quad X=(x,y) \quad A=(0,0)$$

gdje je: $r = \rho(X,A) = \sqrt{(x-0)^2+(y-0)^2}$

Kako je: $\Delta f(X,A) = f(X) - f(A) = f(x,y) - f(0,0) = \frac{xy^2}{x^2+y^2}$

$$df(X,A) = f'_{(x)}(0,0)(x-0) + f'_y(0,0)(y-0) = 0, \text{ to}$$

(x,y) $\omega = \ldots = \cos\varphi \sin^2\varphi$, to $\lim_{\substack{x\to 0\\ y\to 0}} \omega(x,y)$ ne postoji,

jer pri procesu $(x,y)\to(0,0)$, za konstantno φ, $\omega(x,y)$ ima stalno vrijednost $\cos\varphi\sin^2\varphi$ koja zavisi od izbora ugla φ. Zbog toga f-ja nije diferencijabilna u tački (0,0).

* Mogli smo ovo riješiti kao i kod prethodnog zadatka:

$$\underbrace{f(x,y)-f(0,0)}_{0} = \underbrace{f'_{(x)}(0,0)}_{0}(x-0) + \underbrace{f'_y(0,0)}_{0}(y-0) + \omega(x,y)\cdot\sqrt{x^2+y^2} \Rightarrow \omega(x,y) = \frac{f(x,y)}{\sqrt{x^2+y^2}}$$

$$\omega(x,y) = \frac{xy^2}{(x^2+y^2)\sqrt{x^2+y^2}} \Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \left(xy^2\cdot(x^2+y^2)^{-\frac{3}{2}}\right) = \text{prelaskom na: } \left| \begin{matrix} x = \rho\sin\varphi \\ x = \rho\cos\varphi \end{matrix} \right| = \text{ne postoji!}$$

$L = \lim\limits_{\substack{x\to a \\ y\to b}} f(x,y)$ ne postoji, tj: (kod ispitivanja prekidnosti).

ako : 1° postoje L_1 i L_2 (konačni i određeni) a $L_1 \neq L_2$

$L_1 = \lim\limits_{x\to a}(\lim\limits_{y\to b} f(x,y))$, $L_2 = \lim\limits_{x\to b}(\lim\limits_{y\to a} f(x,y))$

2° L zavisi od φ, ako se pređe na polarne koordinate : $x = \rho\cos\varphi$, $y = \rho\sin\varphi$.

3° L zavisi od realnog parametra k, ako se vrši približavanje točki $A=(a,b)$ preko $y=kx$, $y=kx^2$, $y=kx^3$ ili nekoj drugoj u kojoj figuriše k.

4° Dobijemo različite L, ako se točki A približavamo preko različitih nizova točaka.

5° Da je $|f(x,y)| \geq g(x,y)$, gdje $g(x,y)$ divergira u $+\infty$, kad $x\to a$, $y\to b$.

6° Nisu ispunjeni uslovi navedeni u definiciji granične vrijednosti.

Ekstremi f-ja više promjenjivih

1° Nužni uvjeti : Stacionarne točke diferencijabilne f-je $f(x,y)$ nalazimo rješavanjem sistema jednadžbi:

$$f'_x(x,y) = 0 \quad , \quad f'_y(x,y) = 0$$

2° Dovoljni uvjeti : Neka je $X_0(x_0,y_0)$ stac. točka f-je $f(x,y)$, tj., $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$ i neka je:

$r = f_{xx}(x_0,y_0)$, $t = f_{yy}(x_0,y_0)$, $s = f_{xy}(x_0,y_0)$ i

$\Delta = \begin{vmatrix} r & s \\ s & t \end{vmatrix} = rt - s^2$ Tada imamo :

1) Ako je $\Delta > 0$ f-ja ima ekstrem u $X_0(x_0, y_0)$ i to:
Lokalni minimum za $r > 0$ (ili $t > 0$)
Lokalni maksimum za $r < 0$ (ili $t < 0$)

2) Ako je $\Delta < 0$, f-ja nema ekstrema u (x_0, y_0)

3) Ako je $\Delta = 0$ pitanje ekstrema ostaje otvoreno ide se ispitivati pomoću definicije ili diferencijala višeg reda – Tajlorove formule.

Uslovni ekstrem

Uvjetni ekstrem f-je $f(x,y)$ je min. ili max. te f-je dostignut pod uvjetom da su argumenti f-je povezani jednadžbom $\varphi(x,y) = 0$ (jednadžba veze).

Da bi našli uslovni ekstrem derivabilne f-je, uz postojanje derivabilnog odnosa $\varphi(x,y) = 0$ tvorimo Langrangeovu f-ju:

$$F(x,y) = f(x,y) + \lambda \cdot \varphi(x,y), \text{ gdje je}$$

λ neodređen konstantan faktor, i tražimo običan ekstrem te pomoćne f-je: Nužni uvjeti svode se na sistem od tri jednadžbe iz kojih je potrebno odrediti nepoznate x, y, λ.

$$\begin{cases} \dfrac{\partial F}{\partial x} = \dfrac{\partial f}{\partial x} + \lambda \dfrac{\partial \varphi}{\partial x} = 0, \\[2ex] \dfrac{\partial F}{\partial y} = \dfrac{\partial f}{\partial y} + \lambda \dfrac{\partial \varphi}{\partial y} = 0, \\[2ex] \varphi(x,y) = 0 \end{cases} \qquad (1)$$

Sada gledamo predznak drugog diferencijala Lagranžove f-je:

$$d^2F(x,y) = \frac{\partial^2 F}{\partial x^2} dx^2 + 2\frac{\partial^2 F}{\partial x \partial y} dx\,dy + \frac{\partial^2 F}{\partial y^2} dy^2 \text{ za}$$

ispitivani sistem vrijednosti x, y, λ dobiven iz (1), pod uvjetom da su dx i dy vezani jednadžbom:

$\frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = 0, \quad (dx^2+dy^2) \neq 0)$

Ako je: $d^2F < 0 \Rightarrow$ max.

$d^2F > 0 \Rightarrow$ min.

ili u slučaju da je diskriminanta $\Delta = AC - B^2$ za f-ju $F(x,y) = f(x,y) + \lambda \cdot g(x,y)$ u st. točki pozitivna imamo za:

1° $A < 0$ (ili $C < 0$) maximum

2° $A > 0$ (ili $C > 0$) minimum.

Ako je $\Delta \leq 0$ idu dalja ispitivanja!

Primjer: Ispitajmo da li f-ja:

$$z = x^3 + 3x^2y - 15x - 12y \text{ ima ekstrem:}$$

Rj:

$z = x^3 + 3x^2y - 15x - 12y$

Nađimo parcijalne derivacije i tvorimo sistem jednadžbi:

$\frac{\partial z}{\partial x} \equiv 3x^2 + 3y^2 - 15 = 0, \quad \frac{\partial z}{\partial y} \equiv 6xy - 12 = 0$ ili

$$\begin{cases} x^2 + y^2 - 5 = 0, \\ xy - 2 = 0. \end{cases}$$

Rješenjem sistema dobijamo četiri stacionarne točke:

$P_1(1,2)$

$P_2(2,1)$

$P_3(-1,-2)$

$P_4(-2,1)$.

Nađimo derivacije drugog reda:

$\frac{\partial^2 z}{\partial x^2} = 6x, \quad \frac{\partial^2 z}{\partial x \partial y} = 6y, \quad \frac{\partial^2 z}{\partial y^2} = 6x$ i tvorimo

...: $\Delta = AC - B^2$ za svaku stacionarnu tačku.

1) Za tačku P_1 je: $A = \left(\frac{\partial^2 z}{\partial x^2}\right)_{P_1} = 6$

$$B = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_{P_1} = 12$$

$$C = \left(\frac{\partial^2 z}{\partial y^2}\right)_{P_1} = 6$$

$\Delta = AC - B^2 = 36 - 144 < 0$. Znači da u tački P_1 nema ekstrema.

2) Za tačku P_2 je: $A = 12$, $B = 6$, $C = 12$.

$\Delta = 144 - 36 > 0 \Rightarrow$ ekstrem postoji.

$A > 0 \Rightarrow P_2$ je minimum i jednak je vrijednosti f-je kada je $x = 2$, $y = 1$.

$z_{min} = 8 + 6 - 30 - 12 = -28$.

3) Za tačku P_3 je: $A = -6$, $B = -12$, $C = -6$;

$\Delta = 36 - 144 < 0 \Rightarrow$ nema ekstrema.

4) Za tačku P_4 je: $A = -12$, $B = -6$, $C = -12$;

$\Delta = 144 - 36 > 0$, $A < 0 \Rightarrow$ u P_4 je maximum.

$z_{max} = -8 - 6 + 30 + 12 = 28$.

(Z) Nađimo ekstreme f-je:

$z = 6 - 4x - 3y$ pod uvjetom da varijable x i y zadovoljavaju jednačbu: $x^2 + y^2 = 1$.

RJ: Geometrijski se zadatak svodi na traženje najveće i najmanje vrijednosti aplikate z ravnine $z = 6 - 4x - 3y$ za tačke u kojima ravnina siječe valjak $x^2 + y^2 = 1$.

Tvorimo Lagranževu f-ju:

$$F(x,y) = 6-4x-3y+\lambda(x^2+y^2-1).$$

Imamo: $\frac{\partial F}{\partial x} = -4+2\lambda x$, $\frac{\partial F}{\partial y} = -3+2\lambda y$.

Potrebni uslovi daju sistem jednadžbi:

$$\begin{cases} -4+2\lambda x = 0, \\ -3+2\lambda y = 0, \\ x^2+y^2 = 1. \end{cases}$$

kada se ovaj sistem riješi dobijemo da je: $\lambda_1 = \frac{5}{2}$, $x_1 = \frac{4}{5}$, $y_1 = \frac{3}{5}$

$\lambda_2 = \frac{-5}{2}$, $x_2 = -\frac{4}{5}$, $y_2 = -\frac{3}{5}$

kako je: $\frac{\partial^2 F}{\partial x^2} = 2\lambda$, $\frac{\partial^2 F}{\partial x \partial y} = 0$, $\frac{\partial^2 F}{\partial y^2} = 2\lambda$ to je:

$$d^2F = (2\lambda)(dx^2+dy^2)$$

kada je $\lambda = \frac{5}{2}$, $x = \frac{4}{5}$ i $y = \frac{3}{5}$, tada je $d^2F > 0$, i tada f-ja ima uslovni minimum.

kada je $\lambda = -\frac{5}{2}$, $x = -\frac{4}{5}$, $y = -\frac{3}{5}$, tada je $d^2F < 0$ pa u toj tački f-ja ima uslovni maximum.

Na taj način je: $z_{max} = 6+\frac{16}{5}+\frac{9}{5} = 11.$

$z_{min} = 6-\frac{16}{5}-\frac{9}{5} = 1.$